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Some existence theorems for fractional integro-differential equations and inclusions with initial and non-separated boundary conditions

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Abstract

In this paper, we study the existence of solutions for a new class of boundary value problems of nonlinear fractional integro-differential equations and inclusions of arbitrary order with initial and non-separated boundary conditions. In the case of inclusion, the existence results are obtained for convex as well as non-convex multifunctions. Our results rely on the standard tools of fixed point theory and are well illustrated with the aid of examples.

1 Introduction

The subject of fractional calculus has recently been investigated in an extensive manner. The publication of several books, special issues, and a huge number of articles in journals of international repute, exploring numerous aspects of this branch of mathematics, clearly indicates the popularity of the topic. One of the key factors accounting for the utility of the subject is that fractional-order operators are nonlocal in nature in contrast to the integer-order operators and can describe the hereditary properties of many underlying phenomena and processes. Owing to this characteristic, the principles of fractional calculus have played a significant role in improving the modeling techniques for several real world problems [1–4].

Many researchers have focused their attention on fractional differential equations and inclusions, and a variety of interesting and important results concerning existence and uniqueness of solutions, stability properties of solutions, analytic and numerical methods of solutions of these equations have been obtained and the surge for investigating more and more results is still under way. For details and examples, we refer the reader to a series of papers [5–29] and the references therein. Anti-periodic boundary value problems occur in the mathematical modeling of a variety of physical processes and some works have been published in this area, for instance; see [30–34] and the references therein.

In this paper, for $\alpha \in (n-1, n]$, $n \geq 5$, $n \in \mathbb{N}$, $t \in I = [0, T]$, $T > 0$, we investigate the fractional integro-differential equation

$$\begin{aligned} {}^c D^\alpha x(t) = & f(t, x(t), x'(t), x''(t), x'''(t), \phi x(t), \psi x(t), {}^c D^{\mu_1} x(t), {}^c D^{\mu_2} x(t), \dots, {}^c D^{\mu_m} x(t), \\ & {}^c D^{\nu_1} x(t), {}^c D^{\nu_2} x(t), \dots, {}^c D^{\nu_{m'}} x(t), {}^c D^{\xi_1} x(t), {}^c D^{\xi_2} x(t), \dots, {}^c D^{\xi_{m''}} x(t)), \end{aligned} \quad (1.1)$$

and related fractional integro-differential inclusion

$$\begin{aligned} {}^c D^\alpha x(t) \in F(t, x(t), x'(t), x''(t), x'''(t), \phi x(t), \psi x(t), {}^c D^{\mu_1} x(t), {}^c D^{\mu_2} x(t), \dots, {}^c D^{\mu_m} x(t), \\ {}^c D^{\nu_1} x(t), {}^c D^{\nu_2} x(t), \dots, {}^c D^{\nu_{m'}} x(t), {}^c D^{\xi_1} x(t), {}^c D^{\xi_2} x(t), \dots, {}^c D^{\xi_{m''}} x(t)), \end{aligned} \quad (1.2)$$

supplemented with initial boundary conditions

$$\begin{aligned} x^{(4)}(0) = \dots = x^{(n-1)}(0) = 0, \quad ax(0) + bx(T) = 0, \\ {}^c D^p x(0) = -{}^c D^p x(T), \quad {}^c D^q x(0) = -{}^c D^q x(T), \\ {}^c D^\gamma x(0) = -{}^c D^\gamma x(T), \quad 0 < p < 1, 1 < q < 2, 2 < \gamma < 3, a + b \neq 0, a, b \in \mathbb{R}, \end{aligned} \quad (1.3)$$

where ${}^c D$ denotes the Caputo fractional derivative, $f : [0, T] \times \mathbb{R}^{6+m+m'+m''} \rightarrow \mathbb{R}$ is a continuous function, $F : [0, 1] \times \mathbb{R}^{6+m+m'+m''} \rightarrow P(\mathbb{R})$ is a multifunction, $P(\mathbb{R})$ is the family of all non-empty subsets of \mathbb{R} , and the maps ϕ and ψ are defined by

$$\phi x(t) = \int_0^t \gamma(t, s) h_1(t, s, x(s), x'(s), x''(s), x'''(s), {}^c D^{\delta_1} x(s), {}^c D^{\beta_1} x(s), {}^c D^{\theta_1} x(s)) ds$$

and

$$\psi x(t) = \int_0^t \lambda(t, s) h_2(t, s, x(s), x'(s), x''(s), x'''(s), {}^c D^{\delta_2} x(s), {}^c D^{\beta_2} x(s), {}^c D^{\theta_2} x(s)) ds,$$

where $\gamma, \lambda : [0, T] \times [0, T] \rightarrow \mathbb{R}$ and $h_1, h_2 : [0, T] \times [0, T] \times \mathbb{R}^7 \rightarrow \mathbb{R}$ are continuous maps, $0 < \mu_i < 1$ ($1 \leq i \leq m$), $1 < \nu_j < 2$ ($1 \leq j \leq m'$), $2 < \xi_k < 3$ ($1 \leq k \leq m''$), $0 < \delta_i < 1$, $1 < \beta_i < 2$, and $2 < \theta_i < 3$ for $i = 1, 2$.

The paper is organized as follows. In Section 2, we recall some preliminary facts that we used in the sequel. Section 3 deals with the existence result for single-valued initial boundary value problem, while the results for multivalued problem are presented in Section 4. We present some examples illustrating the main results in Section 5.

2 Preliminaries

Let $(X, \|\cdot\|)$ be a normed space, $P(X)$ the set of all non-empty subsets of X , $P_{cl}(X)$ the set of all non-empty closed subsets of X , $P_b(X)$ the set of all non-empty bounded subsets of X , $P_{cp}(X)$ the set of all non-empty compact subsets of X and $P_{cp,c}(X)$ the set of all non-empty compact and convex subsets of X [35]. A multivalued map $G : X \rightarrow P(X)$ is said to be convex (closed) valued whenever $G(x)$ is convex (closed) for all $x \in X$ [35]. The multifunction G is called bounded on bounded sets whenever $G(B) = \bigcup_{x \in B} G(x)$ is bounded subset of X for all $B \in P_b(X)$, that is, $\sup_{x \in B} \{\sup\{|y| : y \in G(x)\}\} < \infty$ for all $B \in P_b(X)$ [35]. Also, the multifunction $G : X \rightarrow P(X)$ is called upper semi-continuous whenever for each $x_0 \in X$ the set $G(x_0)$ is a non-empty closed subset of X , and for every open set N of X containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subseteq N$ [36, 37]. The multifunction $G : X \rightarrow P(X)$ is called compact whenever $G(B)$ is relatively compact for all $B \in P_b(X)$ and also is called completely continuous whenever G is upper semi-continuous and compact [38, 39]. It is well known that a compact multifunction G with non-empty compact valued is upper semi-continuous if and only if G has a closed graph, that is, $x_n \rightarrow x_*$, $y_n \in G(x_n)$ for all n , and $y_n \rightarrow y_*$ imply $y_* \in G(x_*)$ [37]. We say

that $x_0 \in X$ is a fixed point of a multifunction G whenever $x_0 \in G(x_0)$ [40]. Let $T > 0$ and $G : [0, T] \rightarrow P_{cl}(\mathbb{R})$ a multifunction. We say that G is measurable whenever the function $t \mapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$ is measurable for all $y \in \mathbb{R}$ [38, 39].

One can find basic notions of fractional calculus in [1] and [2]. We recall two necessary ones here.

The Riemann-Liouville fractional integral of order $q > 0$ with the lower limit zero for a function $f : [0, \infty) \rightarrow \mathbb{R}$ is defined by $I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(s)}{(t-s)^{1-q}} ds$ for $t > 0$ provided the integral exists.

The Caputo fractional derivative of order $q > 0$ for a function $f \in C^n([0, \infty), \mathbb{R})$ can be written as

$${}^c D^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{q+1-n}} ds = I^{n-q} f^{(n)}(t),$$

where $n-1 < q \leq n$ and $t > 0$.

To define the solution for problems (1.1)-(1.3) and (1.2)-(1.3), we establish the following lemma.

Lemma 2.1 *Let $y \in L^1([0, T], \mathbb{R})$. Then the integral solution of the linear equation*

$${}^c D^\alpha x(t) = y(t) \tag{2.1}$$

subject to the initial boundary conditions (1.3) is given by

$$x(t) = \int_0^T G(t,s)y(s) ds, \tag{2.2}$$

where

$$G(t,s) = \begin{cases} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + G_1(t,s), & \text{if } s \leq t, \\ G_1(t,s), & \text{if } t \leq s, \end{cases}$$

and

$$\begin{aligned} G_1(t,s) = & -\frac{b(T-s)^{\alpha-1}}{(a+b)\Gamma(\alpha)} + \frac{[bT - (a+b)t]\Gamma(2-p)(T-s)^{\alpha-p-1}}{(a+b)\Gamma(\alpha-p)T^{1-p}} \\ & - \frac{[bpT^2 - (a+b)(2Tt - (2-p)t^2)]\Gamma(3-q)(T-s)^{\alpha-q-1}}{2(a+b)(2-p)\Gamma(\alpha-q)T^{2-q}} \\ & - \frac{\Gamma(4-\gamma)(T-s)^{\alpha-\gamma-1}}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \\ & \times (b(-6(q-p) + (2-p)(3-p)q)T^3 \\ & + (a+b)(6(p-q)T^2t + (2-p)(3-p)(-3Tt^2 + (3-q)t^3))). \end{aligned}$$

Proof It is well known that the solution of (2.1) can be written as

$$\begin{aligned} x(t) &= I^\alpha y(t) - b_0 - b_1t - b_2t^2 - b_3t^3 - b_4t^4 - \dots - b_{n-1}t^{n-1} \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}y(s) ds - b_0 - b_1t - b_2t^2 - b_3t^3 - b_4t^4 - \dots - b_{n-1}t^{n-1}, \end{aligned} \tag{2.3}$$

where $b_0, b_1, b_2, b_3, b_4, \dots, b_{n-1} \in \mathbb{R}$ are arbitrary constants. Using the initial conditions $x^{(4)}(0) = \dots = x^{(n-1)}(0) = 0$, we find that $b_4 = \dots = b_{n-1} = 0$. Since ${}^c D^p c = 0$ for all constant c , ${}^c D^p t = \frac{t^{1-p}}{\Gamma(2-p)}$, ${}^c D^p t^2 = \frac{2t^{2-p}}{\Gamma(3-p)}$, ${}^c D^p t^3 = \frac{6t^{3-p}}{\Gamma(4-p)}$, ${}^c D^q t = 0$, ${}^c D^q t^2 = \frac{2t^{2-q}}{\Gamma(3-q)}$, ${}^c D^q t^3 = \frac{6t^{3-q}}{\Gamma(4-q)}$, ${}^c D^\gamma t = 0$, ${}^c D^\gamma t^2 = 0$, ${}^c D^\gamma t^3 = \frac{6t^{3-\gamma}}{\Gamma(4-\gamma)}$, ${}^c D^p I^\alpha y(t) = I^{\alpha-p} y(t)$, ${}^c D^q I^\alpha y(t) = I^{\alpha-q} y(t)$, and ${}^c D^\gamma I^\alpha y(t) = I^{\alpha-\gamma} y(t)$, therefore

$$\begin{aligned} {}^c D^p x(t) &= \frac{1}{\Gamma(\alpha-p)} \int_0^t (t-s)^{\alpha-p-1} y(s) ds - b_1 \frac{t^{1-p}}{\Gamma(2-p)} - b_2 \frac{2t^{2-p}}{\Gamma(3-p)} - b_3 \frac{6t^{3-p}}{\Gamma(4-p)}, \\ {}^c D^q x(t) &= \frac{1}{\Gamma(\alpha-q)} \int_0^t (t-s)^{\alpha-q-1} y(s) ds - b_2 \frac{2t^{2-q}}{\Gamma(3-q)} - b_3 \frac{6t^{3-q}}{\Gamma(4-q)}, \\ {}^c D^\gamma x(t) &= \frac{1}{\Gamma(\alpha-\gamma)} \int_0^t (t-s)^{\alpha-\gamma-1} y(s) ds - b_3 \frac{6t^{3-\gamma}}{\Gamma(4-\gamma)}. \end{aligned}$$

Now using the conditions $ax(0) + bx(T) = 0$, ${}^c D^p x(0) = -{}^c D^p x(T)$, ${}^c D^q x(0) = -{}^c D^q x(T)$, and ${}^c D^\gamma x(0) = -{}^c D^\gamma x(T)$, we obtain

$$\begin{aligned} b_0 &= \frac{b}{(a+b)} \left[\frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} y(s) ds - \frac{\Gamma(2-p)T^p}{\Gamma(\alpha-p)} \int_0^T (T-s)^{\alpha-p-1} y(s) ds \right. \\ &\quad + \frac{p\Gamma(3-q)T^q}{2(2-p)\Gamma(\alpha-q)} \int_0^T (T-s)^{\alpha-q-1} y(s) ds \\ &\quad \left. + \frac{[-6(q-p) + (2-p)(3-p)q]\Gamma(4-\gamma)T^\gamma}{6(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)} \int_0^T (T-s)^{\alpha-\gamma-1} y(s) ds \right], \\ b_1 &= \frac{\Gamma(2-p)}{\Gamma(\alpha-p)T^{1-p}} \int_0^T (T-s)^{\alpha-p-1} y(s) ds \\ &\quad - \frac{\Gamma(3-q)}{(2-p)\Gamma(\alpha-q)T^{1-q}} \int_0^T (T-s)^{\alpha-q-1} y(s) ds \\ &\quad + \frac{(q-p)\Gamma(4-\gamma)}{(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{1-\gamma}} \int_0^T (T-s)^{\alpha-\gamma-1} y(s) ds, \\ b_2 &= \frac{\Gamma(3-q)}{2\Gamma(\alpha-q)T^{2-q}} \int_0^T (T-s)^{\alpha-q-1} y(s) ds \\ &\quad - \frac{\Gamma(4-\gamma)}{2(3-q)\Gamma(\alpha-\gamma)T^{2-\gamma}} \int_0^T (T-s)^{\alpha-\gamma-1} y(s) ds, \\ b_3 &= \frac{\Gamma(4-\gamma)}{6\Gamma(\alpha-\gamma)T^{3-\gamma}} \int_0^T (T-s)^{\alpha-\gamma-1} y(s) ds. \end{aligned}$$

Substituting the values of $b_0, b_1, b_2, b_3, b_4, \dots, b_{n-1}$ in (2.3), we get the solution (2.2). \square

3 Existence results for problem (1.1)-(1.3)

Consider the space $X = \{u : u \in C^3(I)\}$ endowed with the norm

$$\|u\| = \sup_{t \in I} |u(t)| + \sup_{t \in I} |u'(t)| + \sup_{t \in I} |u''(t)| + \sup_{t \in I} |u'''(t)|.$$

Obviously $(X, \|\cdot\|)$ is a Banach space.

We need the following result [40] in the sequel.

Theorem 3.1 Let E be a Banach space, $S : E \rightarrow E$ a completely continuous operator and

$$V = \{x \in E : x = \mu Sx, 0 \leq \mu \leq 1\}$$

a bounded set. Then S has a fixed point in E .

Let us define the operator $T : X \rightarrow X$ by

$$\begin{aligned} (Tx)(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \tilde{f}(s, x(s)) ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \tilde{f}(s, x(s)) ds \\ & + \frac{[bT - (a+b)t]\Gamma(2-p)}{(a+b)\Gamma(\alpha-p)T^{1-p}} \int_0^T (T-s)^{\alpha-p-1} \tilde{f}(s, x(s)) ds \\ & - \frac{[bpT^2 - (a+b)(2Tt - (2-p)t^2)]\Gamma(3-q)}{2(a+b)(2-p)\Gamma(\alpha-q)T^{2-q}} \int_0^T (T-s)^{\alpha-q-1} \tilde{f}(s, x(s)) ds \\ & - \left(\frac{[b(-6(q-p) + (2-p)(3-p)q)T^3]}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right. \\ & \left. + \frac{(a+b)(6(q-p)T^2t + (2-p)(3-p)(-3Tt^2 + (3-q)t^3))\Gamma(4-\gamma)}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right) \\ & \times \int_0^T (T-s)^{\alpha-\gamma-1} \tilde{f}(s, x(s)) ds, \end{aligned}$$

where

$$\begin{aligned} \tilde{f}(s, x(s)) = & f(s, x(s), x'(s), x''(s), x'''(s), \phi x(s), \psi x(s), {}^c D^{\mu_1} x(s), {}^c D^{\mu_2} x(s), \dots, {}^c D^{\mu_m} x(s), \\ & {}^c D^{\nu_1} x(s), {}^c D^{\nu_2} x(s), \dots, {}^c D^{\nu_{m'}} x(s), {}^c D^{\xi_1} x(s), {}^c D^{\xi_2} x(s), \dots, {}^c D^{\xi_{m''}} x(s)). \end{aligned}$$

For the sake of convenience, we set

$$\begin{aligned} M_1 = & \left[\frac{|a| + 2|b|}{|a+b|\Gamma(\alpha+1)} + \frac{(|a| + 2|b|)\Gamma(2-p)}{|a+b|\Gamma(\alpha-p+1)} + \frac{(|b|p + |a+b|(4-p))\Gamma(3-q)}{2|a+b|(2-p)\Gamma(\alpha-q+1)} \right. \\ & + \left(\frac{[|b|(6(q-p) + (2-p)(3-p)q)]}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma+1)} \right. \\ & \left. + \frac{|a+b|(6(q-p) + (2-p)(3-p)(6-q))\Gamma(4-\gamma)}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma+1)} \right) \Big] T^\alpha \\ & + \left[\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(2-p)}{\Gamma(\alpha-p+1)} + \frac{(3-p)\Gamma(3-q)}{(2-p)\Gamma(\alpha-q+1)} \right. \\ & \left. + \frac{[2(q-p) + (2-p)(3-p)(5-q)]\Gamma(4-\gamma)}{2(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma+1)} \right] T^{\alpha-1} \\ & + \left[\frac{1}{\Gamma(\alpha-1)} + \frac{\Gamma(3-q)}{\Gamma(\alpha-q+1)} + \frac{(4-q)\Gamma(4-\gamma)}{(3-q)\Gamma(\alpha-\gamma+1)} \right] T^{\alpha-2} \\ & + \left[\frac{1}{\Gamma(\alpha-2)} + \frac{\Gamma(4-\gamma)}{\Gamma(\alpha-\gamma+1)} \right] T^{\alpha-3}. \end{aligned}$$

Theorem 3.2 The operator $T : X \rightarrow X$ is completely continuous.

Proof First, we show that the operator $T : X \rightarrow X$ is continuous. Let $\{x_n\}$ be a sequence in X with $x_n \rightarrow x_0$ and $0 < \mu_1, \dots, \mu_m < 1$. Then we have

$$\begin{aligned} & \sup_{t \in I} |{}^c D^{\mu_i} x_n(t) - {}^c D^{\mu_i} x_0(t)| \\ &= \sup_{t \in I} \left| \frac{1}{\Gamma(1 - \mu_i)} \int_0^t (t-s)^{-\mu_i} x'_n(s) ds - \frac{1}{\Gamma(1 - \mu_i)} \int_0^t (t-s)^{-\mu_i} x'_0(s) ds \right| \\ &= \sup_{t \in I} \left| \frac{1}{\Gamma(1 - \mu_i)} \int_0^t (t-s)^{-\mu_i} [x'_n(s) - x'_0(s)] ds \right| \\ &\leq \frac{T^{1-\mu_i}}{\Gamma(2 - \mu_i)} \sup_{t \in I} |x'_n(t) - x'_0(t)| \leq \frac{T^{1-\mu_i}}{\Gamma(2 - \mu_i)} \|x_n - x_0\|. \end{aligned}$$

Since $\|x_n - x\| \rightarrow 0$, $\lim_{n \rightarrow \infty} {}^c D^{\mu_i} x_n(t) = {}^c D^{\mu_i} x_0(t)$ uniformly on I . Similarly, $\lim_{n \rightarrow \infty} {}^c D^{\nu_j} x_n(t) = {}^c D^{\nu_j} x_0(t)$ uniformly on I for $1 \leq j \leq m'$, $\lim_{n \rightarrow \infty} {}^c D^{\xi_k} x_n(t) = {}^c D^{\xi_k} x_0(t)$ uniformly on I for $1 \leq k \leq m''$. Also, we get $\lim_{n \rightarrow \infty} {}^c D^{\delta_i} x_n(t) = {}^c D^{\delta_i} x_0(t)$, $\lim_{n \rightarrow \infty} {}^c D^{\beta_i} x_n(t) = {}^c D^{\beta_i} x_0(t)$, and $\lim_{n \rightarrow \infty} {}^c D^{\theta_i} x_n(t) = {}^c D^{\theta_i} x_0(t)$ uniformly on I for $i = 1, 2$. Since

$$\begin{aligned} \|Tx_n - Tx_0\| &= \sup_{t \in I} |Tx_n(t) - Tx_0(t)| + \sup_{t \in I} |(Tx_n)'(t) - (Tx_0)'(t)| \\ &\quad + \sup_{t \in I} |(Tx_n)''(t) - (Tx_0)''(t)| + \sup_{t \in I} |(Tx_n)'''(t) - (Tx_0)'''(t)|, \end{aligned}$$

using the continuity of f , h_1 , h_2 , we get $\|Tx_n - Tx\| \rightarrow 0$. Thus, T is continuous on X . Now, let $\Omega \subseteq X$ be a bounded subset. Then there exists a positive constant $L > 0$ such that $|\tilde{f}(t, x(t))| \leq L$ for all $t \in I$ and $x \in \Omega$. We show that $T\Omega$ is a bounded set. We have

$$\begin{aligned} |(Tx)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |\tilde{f}(s, x(s))| ds + \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |\tilde{f}(s, x(s))| ds \\ &\quad + \frac{|bT - (a+b)t|\Gamma(2-p)}{|a+b|\Gamma(\alpha-p)T^{1-p}} \int_0^T (T-s)^{\alpha-p-1} |\tilde{f}(s, x(s))| ds \\ &\quad + \frac{|bpT^2 - (a+b)(2Tt - (2-p)t^2)|\Gamma(3-q)}{2|a+b|(2-p)\Gamma(\alpha-q)T^{2-q}} \int_0^T (T-s)^{\alpha-q-1} |\tilde{f}(s, x(s))| ds \\ &\quad + \left(\frac{|b(-6(q-p) + (2-p)(3-p)q)T^3}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right. \\ &\quad \left. + \frac{(a+b)(6(q-p)T^2t + (2-p)(3-p)(-3Tt^2 + (3-q)t^3))\Gamma(4-\gamma)}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right) \\ &\quad \times \int_0^T (T-s)^{\alpha-\gamma-1} |\tilde{f}(s, x(s))| ds \\ &\leq \frac{(|a| + 2|b|)LT^\alpha}{|a+b|\Gamma(\alpha+1)} + \frac{(|a| + 2|b|)\Gamma(2-p)LT^\alpha}{|a+b|\Gamma(\alpha-p+1)} \\ &\quad + \frac{(|b|p + |a+b|(4-p))\Gamma(3-q)LT^\alpha}{2|a+b|(2-p)\Gamma(\alpha-q+1)} \\ &\quad + \left(\frac{|b|(6(q-p) + (2-p)(3-p)q)}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma+1)} \right. \\ &\quad \left. + \frac{|a+b|(6(q-p) + (2-p)(3-p)(6-q))\Gamma(4-\gamma)LT^\alpha}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma+1)} \right), \end{aligned}$$

$$\begin{aligned}
|(Tx)'(t)| &\leq \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} |\tilde{f}(s, x(s))| ds \\
&\quad + \frac{\Gamma(2-p)}{\Gamma(\alpha-p)T^{1-p}} \int_0^T (T-s)^{\alpha-p-1} |\tilde{f}(s, x(s))| ds \\
&\quad + \frac{|T-(2-p)t|\Gamma(3-q)}{(2-p)\Gamma(\alpha-q)T^{2-q}} \int_0^T (T-s)^{\alpha-q-1} |\tilde{f}(s, x(s))| ds \\
&\quad + \frac{|2(q-p)T^2 + (2-p)(3-p)(-2Tt + (3-q)t^2)|\Gamma(4-\gamma)}{2(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \\
&\quad \times \int_0^T (T-s)^{\alpha-\gamma-1} |\tilde{f}(s, x(s))| ds \\
&\leq \frac{T^{\alpha-1}L}{\Gamma(\alpha)} + \frac{\Gamma(2-p)LT^{\alpha-1}}{\Gamma(\alpha-p+1)} + \frac{(3-p)\Gamma(3-q)LT^{\alpha-1}}{(2-p)\Gamma(\alpha-q+1)} \\
&\quad + \frac{[2(q-p) + (2-p)(3-p)(5-q)]\Gamma(4-\gamma)LT^{\alpha-1}}{2(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma+1)}, \\
|(Tx)''(t)| &\leq \frac{1}{\Gamma(\alpha-2)} \int_0^t (t-s)^{\alpha-3} |\tilde{f}(s, x(s))| ds \\
&\quad + \frac{\Gamma(3-q)}{\Gamma(\alpha-q)T^{2-q}} \int_0^T (T-s)^{\alpha-q-1} |\tilde{f}(s, x(s))| ds \\
&\quad + \frac{|-T + (3-q)t|\Gamma(4-\gamma)}{(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \int_0^T (T-s)^{\alpha-\gamma-1} |\tilde{f}(s, x(s))| ds \\
&\leq \frac{LT^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{\Gamma(3-q)LT^{\alpha-2}}{\Gamma(\alpha-q+1)} + \frac{(4-q)\Gamma(4-\gamma)LT^{\alpha-2}}{(3-q)\Gamma(\alpha-\gamma+1)},
\end{aligned}$$

and

$$\begin{aligned}
|(Tx)'''(t)| &\leq \frac{1}{\Gamma(\alpha-3)} \int_0^t (t-s)^{\alpha-4} |\tilde{f}(s, x(s))| ds \\
&\quad + \frac{\Gamma(4-\gamma)}{\Gamma(\alpha-\gamma)T^{3-\gamma}} \int_0^T (T-s)^{\alpha-\gamma-1} |\tilde{f}(s, x(s))| ds \\
&\leq \frac{LT^{\alpha-3}}{\Gamma(\alpha-2)} + \frac{\Gamma(4-\gamma)LT^{\alpha-3}}{\Gamma(\alpha-\gamma+1)}
\end{aligned}$$

for all $x \in \Omega$. Hence, we get

$$\begin{aligned}
\|Tx\| &= \sup_{t \in I} |(Tx)(t)| + \sup_{t \in I} |(Tx)'(t)| + \sup_{t \in I} |(Tx)''(t)| + \sup_{t \in I} |(Tx)'''(t)| \\
&\leq \left[\frac{|a| + 2|b|}{|a+b|\Gamma(\alpha+1)} + \frac{(|a| + 2|b|)\Gamma(2-p)}{|a+b|\Gamma(\alpha-p+1)} + \frac{(|b|p + |a+b|(4-p))\Gamma(3-q)}{2|a+b|(2-p)\Gamma(\alpha-q+1)} \right. \\
&\quad + \left(\frac{[|b|(6(q-p) + (2-p)(3-p)q)]}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma+1)} \right. \\
&\quad \left. \left. + \frac{|a+b|(6(q-p) + (2-p)(3-p)(6-q))\Gamma(4-\gamma)}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma+1)} \right) \right] LT^\alpha \\
&\quad + \left[\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(2-p)}{\Gamma(\alpha-p+1)} + \frac{(3-p)\Gamma(3-q)}{(2-p)\Gamma(\alpha-q+1)} \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{[2(q-p) + (2-p)(3-p)(5-q)]\Gamma(4-\gamma)}{2(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma+1)} \Big] LT^{\alpha-1} \\
& + \left[\frac{1}{\Gamma(\alpha-1)} + \frac{\Gamma(3-q)}{\Gamma(\alpha-q+1)} + \frac{(4-q)\Gamma(4-\gamma)}{(3-q)\Gamma(\alpha-\gamma+1)} \right] LT^{\alpha-2} \\
& + \left[\frac{1}{\Gamma(\alpha-2)} + \frac{\Gamma(4-\gamma)}{\Gamma(\alpha-\gamma+1)} \right] LT^{\alpha-3} = M_1 L.
\end{aligned}$$

This implies that the operator T maps bounded sets of X into bounded sets. Now, we prove that the sets $\{Tx : x \in \Omega\}$, $\{(Tx)'\} : x \in \Omega\}$, $\{(Tx)''\} : x \in \Omega\}$, $\{(Tx)'''\} : x \in \Omega\}$ are equicontinuous on I . For $0 \leq t_1 < t_2 \leq T$, we have

$$\begin{aligned}
& |(Tx)(t_2) - (Tx)(t_1)| \\
& = \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2-s)^{\alpha-1} \tilde{f}(s, x(s)) ds - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} \tilde{f}(s, x(s)) ds \right. \\
& \quad - \frac{(t_2-t_1)\Gamma(2-p)}{\Gamma(\alpha-p)T^{1-p}} \int_0^T (T-s)^{\alpha-p-1} \tilde{f}(s, x(s)) ds \\
& \quad + \frac{[2T(t_2-t_1) - (2-p)(t_2^2-t_1^2)]\Gamma(3-q)}{2(2-p)\Gamma(\alpha-q)T^{2-q}} \int_0^T (T-s)^{\alpha-q-1} \tilde{f}(s, x(s)) ds \\
& \quad - \frac{[6(q-p)T^2(t_2-t_1) + (2-p)(3-p)(-3T(t_2^2-t_1^2) + (3-q)(t_2^3-t_1^3))]\Gamma(4-\gamma)}{6(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \\
& \quad \times \left. \int_0^T (T-s)^{\alpha-\gamma-1} \tilde{f}(s, x(s)) ds \right| \\
& \leq \frac{L}{\Gamma(\alpha)} \int_0^{t_1} [(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}] ds + \frac{L}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} ds \\
& \quad + \frac{(t_2-t_1)\Gamma(2-p)LT^{\alpha-1}}{\Gamma(\alpha-p+1)} + \frac{[2T(t_2-t_1) + (2-p)(t_2^2-t_1^2)]\Gamma(3-q)LT^{\alpha-2}}{2(2-p)\Gamma(\alpha-q+1)} \\
& \quad + \frac{[6(q-p)T^2(t_2-t_1) + (2-p)(3-p)(3T(t_2^2-t_1^2) + (3-q)(t_2^3-t_1^3))]\Gamma(4-\gamma)LT^{\alpha-3}}{6(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma+1)} \\
& \leq \frac{L}{\Gamma(\alpha+1)} (t_2^\alpha - t_1^\alpha) \\
& \quad + \frac{(t_2-t_1)\Gamma(2-p)LT^{\alpha-1}}{\Gamma(\alpha-p+1)} + \frac{[2T(t_2-t_1) + (2-p)(t_2^2-t_1^2)]\Gamma(3-q)LT^{\alpha-2}}{2(2-p)\Gamma(\alpha-q+1)} \\
& \quad + \frac{[6(q-p)T^2(t_2-t_1) + (2-p)(3-p)(3T(t_2^2-t_1^2) + (3-q)(t_2^3-t_1^3))]\Gamma(4-\gamma)LT^{\alpha-3}}{6(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma+1)}.
\end{aligned}$$

In a similar manner, one can find that

$$\begin{aligned}
& |(Tx)'(t_2) - (Tx)'(t_1)| \leq \frac{L}{\Gamma(\alpha)} (t_2^{\alpha-1} - t_1^{\alpha-1}) + \frac{(t_2-t_1)\Gamma(3-q)LT^{\alpha-2}}{\Gamma(\alpha-q+1)} \\
& \quad + \frac{[2T(t_2-t_1) + (3-q)(t_2^2-t_1^2)]\Gamma(4-\gamma)LT^{\alpha-3}}{2(3-q)\Gamma(\alpha-\gamma+1)}, \\
& |(Tx)''(t_2) - (Tx)''(t_1)| \leq \frac{L}{\Gamma(\alpha-1)} (t_2^{\alpha-2} - t_1^{\alpha-2}) + \frac{(t_2-t_1)\Gamma(4-\gamma)LT^{\alpha-3}}{\Gamma(\alpha-\gamma+1)},
\end{aligned}$$

and

$$|(Tx)'''(t_2) - (Tx)'''(t_1)| \leq \frac{L}{\Gamma(\alpha - 2)}(t_2^{\alpha-3} - t_1^{\alpha-3}).$$

Clearly the right-hand sides of the above inequalities tend to zero as $t_2 \rightarrow t_1$. So T is completely continuous. This completes the proof. \square

Theorem 3.3 Assume that there exist positive constants $d_0 > 0$, $d_i \geq 0$ ($1 \leq i \leq 6$), $\zeta_i \geq 0$ ($1 \leq i \leq m$), $\eta_j \geq 0$ ($1 \leq j \leq m'$), $\tau_k \geq 0$ ($1 \leq k \leq m''$), $l_{i1}, l_{i2} \geq 0$ ($1 \leq i \leq 7$), $c_{01}, c_{02} > 0$ such that

$$\begin{aligned} & |f(t, x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2, \dots, y_m, z_1, z_2, \dots, z_{m'}, w_1, w_2, \dots, w_{m''})| \\ & \leq d_0 + d_1|x_1| + d_2|x_2| + d_3|x_3| + d_4|x_4| + d_5|x_5| + d_6|x_6| \\ & \quad + \sum_{i=1}^m \zeta_i|y_i| + \sum_{j=1}^{m'} \eta_j|z_j| + \sum_{k=1}^{m''} \tau_k|w_k| \end{aligned}$$

for all $t \in I$ and $x_1, \dots, x_6, y_1, \dots, y_m, z_1, \dots, z_{m'}, \tau_1, \dots, \tau_{m''} \in \mathbb{R}$ and

$$|h_j(t, s, u_1, u_2, u_3, u_4, u_5, u_6, u_7)| \leq c_{0j} + \sum_{i=1}^7 l_{ij}|u_i|$$

for $j = 1, 2$, all $t, s \in I$, and all $u_1, \dots, u_7 \in \mathbb{R}$. In addition, assume that

$$\begin{aligned} M'_1 = M_1 & \left[d_1 + d_2 + d_3 + d_4 + d_5\gamma_0 \left(l_{11} + l_{21} + l_{31} + l_{41} \right. \right. \\ & \quad \left. \left. + l_{51} \frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} + l_{61} \frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + l_{71} \frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)} \right) \right. \\ & \quad \left. + d_6\lambda_0 \left(l_{12} + l_{22} + l_{32} + l_{42} + l_{52} \frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} + l_{62} \frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + l_{72} \frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)} \right) \right. \\ & \quad \left. + \sum_{i=1}^m \zeta_i \frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} + \sum_{j=1}^{m'} \eta_j \frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} + \sum_{k=1}^{m''} \tau_k \frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)} \right] < 1, \end{aligned}$$

where $\gamma_0 = \sup_{t \in I} \int_0^t |\gamma(t, s)| ds$ and $\lambda_0 = \sup_{t \in I} \int_0^t |\lambda(t, s)| ds$. Then problem (1.1)-(1.3) has at least one solution.

Proof In view of Theorem 3.2, the operator $T : X \rightarrow X$ is completely continuous. Next we show that the set $V = \{x \in X : x = \mu Tx, 0 \leq \mu \leq 1\}$ is bounded. Let $x \in V$ and $t \in I$. Then we have

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t \mu(t-s)^{\alpha-1} \tilde{f}(s, x(s)) ds \\ & \quad - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T \mu(T-s)^{\alpha-1} \tilde{f}(s, x(s)) ds \\ & \quad + \frac{[bT - (a+b)t]\Gamma(2-p)}{(a+b)\Gamma(\alpha-p)T^{1-p}} \int_0^T \mu(T-s)^{\alpha-p-1} \tilde{f}(s, x(s)) ds \end{aligned}$$

$$\begin{aligned}
& - \frac{[bpT^2 - (a+b)(2Tt - (2-p)t^2)]\Gamma(3-q)}{2(a+b)(2-p)\Gamma(\alpha-q)T^{2-q}} \int_0^T \mu(T-s)^{\alpha-q-1} \tilde{f}(s, x(s)) ds \\
& - \left(\frac{[b(-6(q-p) + (2-p)(3-p)q)T^3]}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right. \\
& \left. + \frac{(a+b)(6(q-p)T^2t + (2-p)(3-p)(-3Tt^2 + (3-q)t^3))\Gamma(4-\gamma)}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right) \\
& \times \int_0^T \mu(T-s)^{\alpha-\gamma-1} \tilde{f}(s, x(s)) ds, \\
x'(t) &= \frac{1}{\Gamma(\alpha-1)} \int_0^t \mu(t-s)^{\alpha-2} \tilde{f}(s, x(s)) ds \\
& - \frac{\Gamma(2-p)}{\Gamma(\alpha-p)T^{1-p}} \int_0^T \mu(T-s)^{\alpha-p-1} \tilde{f}(s, x(s)) ds \\
& + \frac{[T - (2-p)t]\Gamma(3-q)}{(2-p)\Gamma(\alpha-q)T^{2-q}} \int_0^T \mu(T-s)^{\alpha-q-1} \tilde{f}(s, x(s)) ds \\
& - \frac{[2(q-p)T^2 + (2-p)(3-p)(-2Tt + (3-q)t^2)]\Gamma(4-\gamma)}{2(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \\
& \times \int_0^T \mu(T-s)^{\alpha-\gamma-1} \tilde{f}(s, x(s)) ds, \\
x''(t) &= \frac{1}{\Gamma(\alpha-2)} \int_0^t \mu(t-s)^{\alpha-3} \tilde{f}(s, x(s)) ds \\
& - \frac{\Gamma(3-q)}{\Gamma(\alpha-q)T^{2-q}} \int_0^T \mu(T-s)^{\alpha-q-1} \tilde{f}(s, x(s)) ds \\
& - \frac{[-T + (3-q)t]\Gamma(4-\gamma)}{(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \int_0^T \mu(T-s)^{\alpha-\gamma-1} \tilde{f}(s, x(s)) ds,
\end{aligned}$$

and $x'''(t) = \frac{1}{\Gamma(\alpha-3)} \int_0^t \mu(t-s)^{\alpha-4} \tilde{f}(s, x(s)) ds - \frac{\Gamma(4-\gamma)}{\Gamma(\alpha-\gamma)T^{3-\gamma}} \int_0^T \mu(T-s)^{\alpha-\gamma-1} \tilde{f}(s, x(s)) ds$. Thus, we get

$$\begin{aligned}
|x(t)| &= \mu |Tx(t)| \\
&\leq \left[d_0 + d_1 \|x\| + d_2 \|x\| + d_3 \|x\| + d_4 \|x\| \right. \\
&\quad + d_5 \gamma_0 \left(c_{01} + l_{11} \|x\| + l_{21} \|x\| + l_{31} \|x\| + l_{41} \|x\| \right. \\
&\quad \left. + l_{51} \frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} \|x\| + l_{61} \frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} \|x\| + l_{71} \frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)} \|x\| \right) \\
&\quad + d_6 \lambda_0 \left(c_{02} + l_{12} \|x\| + l_{22} \|x\| + l_{32} \|x\| + l_{42} \|x\| \right. \\
&\quad \left. + l_{52} \frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} \|x\| + l_{62} \frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} \|x\| + l_{72} \frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)} \|x\| \right) \\
&\quad \left. + \sum_{i=1}^m \zeta_i \frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} \|x\| + \sum_{j=1}^{m'} \eta_j \frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} \|x\| + \sum_{k=1}^{m''} \tau_k \frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)} \|x\| \right] \\
&\quad \times \left[\frac{(|a| + 2|b|)T^\alpha}{|a+b|\Gamma(\alpha+1)} + \frac{(|a| + 2|b|)\Gamma(2-p)T^\alpha}{|a+b|\Gamma(\alpha-p+1)} \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{(|b|p + |a + b|(4 - p))\Gamma(3 - q)T^\alpha}{2|a + b|(2 - p)\Gamma(\alpha - q + 1)} \\
& + \left(\frac{[|b|(6(q - p) + (2 - p)(3 - p)q)]}{6|a + b|(2 - p)(3 - p)(3 - q)\Gamma(\alpha - \gamma + 1)} \right. \\
& \left. + \frac{|a + b|(6(q - p) + (2 - p)(3 - p)(6 - q))\Gamma(4 - \gamma)T^\alpha}{6|a + b|(2 - p)(3 - p)(3 - q)\Gamma(\alpha - \gamma + 1)} \right), \\
|x'(t)| &= \mu |(Tx)'(t)| \\
&\leq \left[d_0 + d_1 \|x\| + d_2 \|x\| + d_3 \|x\| + d_4 \|x\| \right. \\
&\quad + d_5 \gamma_0 \left(c_{01} + l_{11} \|x\| + l_{21} \|x\| + l_{31} \|x\| + l_{41} \|x\| \right. \\
&\quad \left. + l_{51} \frac{T^{1-\delta_1}}{\Gamma(2 - \delta_1)} \|x\| + l_{61} \frac{T^{2-\beta_1}}{\Gamma(3 - \beta_1)} \|x\| + l_{71} \frac{T^{3-\theta_1}}{\Gamma(4 - \theta_1)} \|x\| \right) \\
&\quad + d_6 \lambda_0 \left(c_{02} + l_{12} \|x\| + l_{22} \|x\| + l_{32} \|x\| + l_{42} \|x\| \right. \\
&\quad \left. + l_{52} \frac{T^{1-\delta_2}}{\Gamma(2 - \delta_2)} \|x\| + l_{62} \frac{T^{2-\beta_2}}{\Gamma(3 - \beta_2)} \|x\| + l_{72} \frac{T^{3-\theta_2}}{\Gamma(4 - \theta_2)} \|x\| \right) \\
&\quad + \sum_{i=1}^m \zeta_i \frac{T^{1-\mu_i}}{\Gamma(2 - \mu_i)} \|x\| + \sum_{j=1}^{m'} \eta_j \frac{T^{2-\nu_j}}{\Gamma(3 - \nu_j)} \|x\| + \sum_{k=1}^{m''} \tau_k \frac{T^{3-\xi_k}}{\Gamma(4 - \xi_k)} \|x\| \Big] \\
&\quad \times \left[\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{\Gamma(2 - p)T^{\alpha-1}}{\Gamma(\alpha - p + 1)} + \frac{(3 - p)\Gamma(3 - q)T^{\alpha-1}}{(2 - p)\Gamma(\alpha - q + 1)} \right. \\
&\quad \left. + \frac{[2(q - p) + (2 - p)(3 - p)(5 - q)]\Gamma(4 - \gamma)T^{\alpha-1}}{2(2 - p)(3 - p)(3 - q)\Gamma(\alpha - \gamma + 1)} \right], \\
|x''(t)| &= \mu |(Tx)''(t)| \\
&\leq \left[d_0 + d_1 \|x\| + d_2 \|x\| + d_3 \|x\| + d_4 \|x\| \right. \\
&\quad + d_5 \gamma_0 \left(c_{01} + l_{11} \|x\| + l_{21} \|x\| + l_{31} \|x\| + l_{41} \|x\| \right. \\
&\quad + l_{51} \frac{T^{1-\delta_1}}{\Gamma(2 - \delta_1)} \|x\| + l_{61} \frac{T^{2-\beta_1}}{\Gamma(3 - \beta_1)} \|x\| + l_{71} \frac{T^{3-\theta_1}}{\Gamma(4 - \theta_1)} \|x\| \Big) \\
&\quad + d_6 \lambda_0 \left(c_{02} + l_{12} \|x\| + l_{22} \|x\| + l_{32} \|x\| + l_{42} \|x\| \right. \\
&\quad + l_{52} \frac{T^{1-\delta_2}}{\Gamma(2 - \delta_2)} \|x\| + l_{62} \frac{T^{2-\beta_2}}{\Gamma(3 - \beta_2)} \|x\| + l_{72} \frac{T^{3-\theta_2}}{\Gamma(4 - \theta_2)} \|x\| \Big) \\
&\quad + \sum_{i=1}^m \zeta_i \frac{T^{1-\mu_i}}{\Gamma(2 - \mu_i)} \|x\| + \sum_{j=1}^{m'} \eta_j \frac{T^{2-\nu_j}}{\Gamma(3 - \nu_j)} \|x\| + \sum_{k=1}^{m''} \tau_k \frac{T^{3-\xi_k}}{\Gamma(4 - \xi_k)} \|x\| \Big] \\
&\quad \times \left[\frac{T^{\alpha-2}}{\Gamma(\alpha - 1)} + \frac{\Gamma(3 - q)T^{\alpha-2}}{\Gamma(\alpha - q + 1)} + \frac{(4 - q)\Gamma(4 - \gamma)T^{\alpha-2}}{(3 - q)\Gamma(\alpha - \gamma + 1)} \right],
\end{aligned}$$

and

$$\begin{aligned}
 |x'''(t)| &= \mu |(Tx)'''(t)| \\
 &\leq \left[d_0 + d_1 \|x\| + d_2 \|x\| + d_3 \|x\| + d_4 \|x\| \right. \\
 &\quad + d_5 \gamma_0 \left(c_{01} + l_{11} \|x\| + l_{21} \|x\| + l_{31} \|x\| + l_{41} \|x\| \right. \\
 &\quad \left. + l_{51} \frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} \|x\| + l_{61} \frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} \|x\| + l_{71} \frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)} \|x\| \right) \\
 &\quad + d_6 \lambda_0 \left(c_{02} + l_{12} \|x\| + l_{22} \|x\| + l_{32} \|x\| + l_{42} \|x\| \right. \\
 &\quad \left. + l_{52} \frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} \|x\| + l_{62} \frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} \|x\| + l_{72} \frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)} \|x\| \right) \\
 &\quad + \sum_{i=1}^m \zeta_i \frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} \|x\| + \sum_{j=1}^{m'} \eta_j \frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} \|x\| + \sum_{k=1}^{m''} \tau_k \frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)} \|x\| \left. \right] \\
 &\quad \times \left[\frac{T^{\alpha-3}}{\Gamma(\alpha-2)} + \frac{\Gamma(4-\gamma) T^{\alpha-3}}{\Gamma(\alpha-\gamma+1)} \right].
 \end{aligned}$$

This implies that

$$\begin{aligned}
 \|x\| &\leq M_1 \left[d_1 + d_2 + d_3 + d_4 + d_5 \gamma_0 \left(l_{11} + l_{21} + l_{31} + l_{41} \right. \right. \\
 &\quad \left. + l_{51} \frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} + l_{61} \frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + l_{71} \frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)} \right) \\
 &\quad + d_6 \lambda_0 \left(l_{12} + l_{22} + l_{32} + l_{42} + l_{52} \frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} + l_{62} \frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + l_{72} \frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)} \right) \\
 &\quad \left. + \sum_{i=1}^m \zeta_i \frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} + \sum_{j=1}^{m'} \eta_j \frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} + \sum_{k=1}^{m''} \tau_k \frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)} \right] \|x\| \\
 &\quad + M_1 (d_0 + d_5 \gamma_0 c_{01} + d_6 \lambda_0 c_{02})
 \end{aligned}$$

and so $\|x\| \leq \frac{M_1(d_0+d_5\gamma_0c_{01}+d_6\lambda_0c_{02})}{1-M_1}$. Thus, the set V is bounded. Hence it follows by Theorem 3.1 that the operator T has at least one fixed point, which in turn implies that problem (1.1)-(1.3) has a solution. \square

Theorem 3.4 Assume that $f : I \times \mathbb{R}^{6+m+m'+m''} \rightarrow \mathbb{R}$ and $h_1, h_2 : I \times I \times \mathbb{R}^7 \rightarrow \mathbb{R}$ are continuous functions and there exist constants $n_i \geq 0$ ($1 \leq i \leq 6$), $k_i \geq 0$ ($1 \leq i \leq m$), $k'_j \geq 0$ ($1 \leq j \leq m'$), $k''_k \geq 0$ ($1 \leq k \leq m''$), $e_{i1}, e_{i2} \geq 0$ ($1 \leq i \leq 7$) such that

$$\begin{aligned}
 &|f(t, x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2, \dots, y_m, z_1, z_2, \dots, z_{m'}, w_1, w_2, \dots, w_{m''}) \\
 &\quad - f(t, x'_1, x'_2, x'_3, x'_4, x'_5, x'_6, y'_1, y'_2, \dots, y'_m, z'_1, z'_2, \dots, z'_{m'}, w'_1, w'_2, \dots, w'_{m''})| \\
 &\leq n_1 |x_1 - x'_1| + n_2 |x_2 - x'_2| + n_3 |x_3 - x'_3|
 \end{aligned}$$

$$+ n_4 |x_4 - x'_4| + n_5 |x_5 - x'_5| + n_6 |x_6 - x'_6| \\ + \sum_{i=1}^m k_i |y_i - y'_i| + \sum_{j=1}^{m'} k'_j |z_j - z'_j| + \sum_{k=1}^{m''} k''_k |w_k - w'_k|$$

for all $x_1, \dots, x_6, x'_1, \dots, x'_6, y_1, \dots, y_m, y'_1, \dots, y'_m, z_1, \dots, z_{m'}, z'_1, \dots, z'_{m'}, w_1, \dots, w_{m''}, w'_1, \dots, w'_{m''} \in \mathbb{R}$, and $t \in I$, and also

$$|h_j(t, s, u_1, u_2, u_3, u_4, u_5, u_6, u_7) - h_j(t, s, u'_1, u'_2, u'_3, u'_4, u'_5, u'_6, u'_7)| \leq \sum_{i=1}^7 e_{ij} |u_i - u'_i|$$

for $j = 1, 2, t, s \in I$, and $u_1, \dots, u_7, u'_1, \dots, u'_7 \in \mathbb{R}$. In addition, suppose that

$$\Delta = M_1 \left[n_1 + n_2 + n_3 + n_4 + n_5 \gamma_0 \left(e_{11} + e_{21} + e_{31} + e_{41} \right. \right. \\ \left. \left. + e_{51} \frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} + e_{61} \frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + e_{71} \frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)} \right) \right. \\ \left. + n_6 \lambda_0 \left(e_{12} + e_{22} + e_{32} + e_{42} + e_{52} \frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} + e_{62} \frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + e_{72} \frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)} \right) \right. \\ \left. + \sum_{i=1}^m k_i \frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} + \sum_{j=1}^{m'} k'_j \frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} + \sum_{k=1}^{m''} k''_k \frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)} \right] < 1.$$

Then problem (1.1)-(1.3) has a unique solution.

Proof Set $N = \sup_{0 \leq t \leq T} |f(t, 0, 0, \dots, 0)| < \infty$, $\kappa_j = \sup_{0 \leq t, s \leq T} |h_j(t, s, 0, 0, \dots, 0)| < \infty$ for $j = 1, 2$, and choose $r \geq \frac{(N+n_5\gamma_0\kappa_1+n_6\lambda_0\kappa_2)M_1}{1-\Delta}$. We show that $T(B_r) \subseteq B_r$, where $B_r = \{x \in X : \|x\| \leq r\}$. Let $x \in B_r$. Then we have

$$|(Tx)(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |\tilde{f}(s, x(s))| ds + \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |\tilde{f}(s, x(s))| ds \\ + \frac{|bT - (a+b)t|\Gamma(2-p)}{|a+b|\Gamma(\alpha-p)T^{1-p}} \int_0^T (T-s)^{\alpha-p-1} |\tilde{f}(s, x(s))| ds \\ + \frac{|bpT^2 - (a+b)(2Tt - (2-p)t^2)|\Gamma(3-q)}{2|a+b|(2-p)\Gamma(\alpha-q)T^{2-q}} \int_0^T (T-s)^{\alpha-q-1} |\tilde{f}(s, x(s))| ds \\ + \left(\frac{|b(-6(q-p) + (2-p)(3-p)q)T^3}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right. \\ \left. + \frac{(a+b)(6(q-p)T^2t + (2-p)(3-p)(-3Tt^2 + (3-q)t^3))\Gamma(4-\gamma)}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right) \\ \times \int_0^T (T-s)^{\alpha-\gamma-1} |\tilde{f}(s, x(s))| ds \\ \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [|\tilde{f}(s, x(s)) - f(s, 0, 0, \dots, 0)| + |f(s, 0, 0, \dots, 0)|] ds \\ + \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} [|\tilde{f}(s, x(s)) - f(s, 0, 0, \dots, 0)|$$

$$\begin{aligned}
& + |f(s, 0, 0, \dots, 0)|] ds \\
& + \frac{|bT - (a+b)t|\Gamma(2-p)}{|a+b|\Gamma(\alpha-p)T^{1-p}} \int_0^T (T-s)^{\alpha-p-1} [|\tilde{f}(s, x(s)) - f(s, 0, 0, \dots, 0)| \\
& + |f(s, 0, 0, \dots, 0)|] ds \\
& + \frac{|bpT^2 - (a+b)(2Tt - (2-p)t^2)|\Gamma(3-q)}{2|a+b|(2-p)\Gamma(\alpha-q)T^{2-q}} \\
& \times \int_0^T (T-s)^{\alpha-q-1} [|\tilde{f}(s, x(s)) - f(s, 0, 0, \dots, 0)| + |f(s, 0, 0, \dots, 0)|] ds \\
& + \left(\frac{|b(-6(q-p) + (2-p)(3-p)q)T^3}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right. \\
& + \left. \frac{(a+b)(6(q-p)T^2t + (2-p)(3-p)(-3Tt^2 + (3-q)t^3))\Gamma(4-\gamma)}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right) \\
& \times \int_0^T (T-s)^{\alpha-\gamma-1} [|\tilde{f}(s, x(s)) - f(s, 0, 0, \dots, 0)| + |f(s, 0, 0, \dots, 0)|] ds \\
& \leq \left[\left(n_1 + n_2 + n_3 + n_4 + n_5\gamma_0 \left(e_{11} + e_{21} + e_{31} + e_{41} \right. \right. \right. \\
& + e_{51} \frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} + e_{61} \frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + e_{71} \frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)} \Bigg) \\
& + n_6\lambda_0 \left(e_{12} + e_{22} + e_{32} + e_{42} + e_{52} \frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} + e_{62} \frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + e_{72} \frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)} \right) \\
& + \sum_{i=1}^m k_i \frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} + \sum_{j=1}^{m'} k'_j \frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} + \sum_{k=1}^{m''} k''_k \frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)} \Bigg) r \\
& + N + n_5\gamma_0\kappa_1 + n_6\lambda_0\kappa_2 \Bigg] \\
& \times \left[\frac{(|a| + 2|b|)T^\alpha}{|a+b|\Gamma(\alpha+1)} + \frac{(|a| + 2|b|)\Gamma(2-p)T^\alpha}{|a+b|\Gamma(\alpha-p+1)} \right. \\
& + \frac{(|b|p + |a+b|(4-p))\Gamma(3-q)T^\alpha}{2|a+b|(2-p)\Gamma(\alpha-q+1)} \\
& + \left(\frac{|b|(6(q-p) + (2-p)(3-p)q)}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma+1)} \right. \\
& + \left. \left. \frac{|a+b|(6(q-p) + (2-p)(3-p)(6-q))\Gamma(4-\gamma)T^\alpha}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma+1)} \right) \right].
\end{aligned}$$

In a similar way, we can obtain

$$\begin{aligned}
& |(Tx)'(t)| \\
& \leq \left[\left(n_1 + n_2 + n_3 + n_4 + n_5\gamma_0 \left(e_{11} + e_{21} + e_{31} + e_{41} \right. \right. \right. \\
& + e_{51} \frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} + e_{61} \frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + e_{71} \frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)} \Bigg)
\end{aligned}$$

$$\begin{aligned}
& + n_6 \lambda_0 \left(e_{12} + e_{22} + e_{32} + e_{42} + e_{52} \frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} + e_{62} \frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + e_{72} \frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)} \right) \\
& + \sum_{i=1}^m k_i \frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} + \sum_{j=1}^{m'} k'_j \frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} + \sum_{k=1}^{m''} k''_k \frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)} \Big) r + N + n_5 \gamma_0 \kappa_1 + n_6 \lambda_0 \kappa_2 \Big] \\
& \times \left[\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{\Gamma(2-p)T^{\alpha-1}}{\Gamma(\alpha-p+1)} + \frac{(3-p)\Gamma(3-q)T^{\alpha-1}}{(2-p)\Gamma(\alpha-q+1)} \right. \\
& \left. + \frac{[2(q-p) + (2-p)(3-p)(5-q)]\Gamma(4-\gamma)T^{\alpha-1}}{2(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma+1)} \right], \\
& |(Tx)''(t)| \\
& \leq \left[\left(n_1 + n_2 + n_3 + n_4 + n_5 \gamma_0 \left(e_{11} + e_{21} + e_{31} + e_{41} \right. \right. \right. \\
& \quad \left. \left. + e_{51} \frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} + e_{61} \frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + e_{71} \frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)} \right) \right. \\
& \quad \left. + n_6 \lambda_0 \left(e_{12} + e_{22} + e_{32} + e_{42} + e_{52} \frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} + e_{62} \frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + e_{72} \frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)} \right) \right. \\
& \quad \left. + \sum_{i=1}^m k_i \frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} + \sum_{j=1}^{m'} k'_j \frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} + \sum_{k=1}^{m''} k''_k \frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)} \right) r + N + n_5 \gamma_0 \kappa_1 + n_6 \lambda_0 \kappa_2 \Big] \\
& \times \left[\frac{T^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{\Gamma(3-q)T^{\alpha-2}}{\Gamma(\alpha-q+1)} + \frac{(4-q)\Gamma(4-\gamma)T^{\alpha-2}}{(3-q)\Gamma(\alpha-\gamma+1)} \right],
\end{aligned}$$

and

$$\begin{aligned}
& |(Tx)'''(t)| \\
& \leq \left[\left(n_1 + n_2 + n_3 + n_4 + n_5 \gamma_0 \left(e_{11} + e_{21} + e_{31} + e_{41} \right. \right. \right. \\
& \quad \left. \left. + e_{51} \frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} + e_{61} \frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + e_{71} \frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)} \right) \right. \\
& \quad \left. + n_6 \lambda_0 \left(e_{12} + e_{22} + e_{32} + e_{42} + e_{52} \frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} + e_{62} \frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + e_{72} \frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)} \right) \right. \\
& \quad \left. + \sum_{i=1}^m k_i \frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} + \sum_{j=1}^{m'} k'_j \frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} + \sum_{k=1}^{m''} k''_k \frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)} \right) r \\
& \quad \left. + N + n_5 \gamma_0 \kappa_1 + n_6 \lambda_0 \kappa_2 \right] \left[\frac{T^{\alpha-3}}{\Gamma(\alpha-2)} + \frac{\Gamma(4-\gamma)T^{\alpha-3}}{\Gamma(\alpha-\gamma+1)} \right].
\end{aligned}$$

Hence,

$$\begin{aligned}
\|Tx\| \leq & \left[\left(n_1 + n_2 + n_3 + n_4 + n_5 \gamma_0 \left(e_{11} + e_{21} + e_{31} + e_{41} \right. \right. \right. \\
& \quad \left. \left. + e_{51} \frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} + e_{61} \frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + e_{71} \frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)} \right) \right.
\end{aligned}$$

$$\begin{aligned}
& + n_6 \lambda_0 \left(e_{12} + e_{22} + e_{32} + e_{42} + e_{52} \frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} + e_{62} \frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + e_{72} \frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)} \right) \\
& + \sum_{i=1}^m k_i \frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} + \sum_{j=1}^{m'} k'_j \frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} + \sum_{k=1}^{m''} k''_k \frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)} \Bigg) r \\
& + N + n_5 \gamma_0 \kappa_1 + n_6 \lambda_0 \kappa_2 \Bigg] M_1 \leq r
\end{aligned}$$

and so $\|Tx\| \leq r$. Also, we have

$$\begin{aligned}
& |(Tu)(t) - (Tv)(t)| \\
& \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [|\tilde{f}(s, u(s)) - \tilde{f}(s, v(s))|] ds \\
& + \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} [|\tilde{f}(s, u(s)) - \tilde{f}(s, v(s))|] ds \\
& + \frac{|bT - (a+b)t|\Gamma(2-p)}{|a+b|\Gamma(\alpha-p)T^{1-p}} \int_0^T (T-s)^{\alpha-p-1} [|\tilde{f}(s, u(s)) - \tilde{f}(s, v(s))|] ds \\
& + \frac{|bpT^2 - (a+b)(2Tt - (2-p)t^2)|\Gamma(3-q)}{2|a+b|(2-p)\Gamma(\alpha-q)T^{2-q}} \\
& \times \int_0^T (T-s)^{\alpha-q-1} [|\tilde{f}(s, u(s)) - \tilde{f}(s, v(s))|] ds \\
& + \left(\frac{|b(-6(q-p) + (2-p)(3-p)q)T^3}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right. \\
& + \left. \frac{(a+b)(6(q-p)T^2t + (2-p)(3-p)(-3Tt^2 + (3-q)t^3))\Gamma(4-\gamma)}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right) \\
& \times \int_0^T (T-s)^{\alpha-\gamma-1} [|\tilde{f}(s, u(s)) - \tilde{f}(s, v(s))|] ds \\
& \leq \left[n_1 + n_2 + n_3 + n_4 + n_5 \gamma_0 \left(e_{11} + e_{21} + e_{31} + e_{41} \right. \right. \\
& + e_{51} \frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} + e_{61} \frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + e_{71} \frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)} \Bigg) \\
& + n_6 \lambda_0 \left(e_{12} + e_{22} + e_{32} + e_{42} + e_{52} \frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} + e_{62} \frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + e_{72} \frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)} \right) \\
& + \sum_{i=1}^m k_i \frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} + \sum_{j=1}^{m'} k'_j \frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} + \sum_{k=1}^{m''} k''_k \frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)} \Bigg] \\
& \times \left[\frac{(|a| + 2|b|)T^\alpha}{|a+b|\Gamma(\alpha+1)} + \frac{(|a| + 2|b|)\Gamma(2-p)T^\alpha}{|a+b|\Gamma(\alpha-p+1)} + \frac{(|b|p + |a+b|(4-p))\Gamma(3-q)T^\alpha}{2|a+b|(2-p)\Gamma(\alpha-q+1)} \right. \\
& + \left(\frac{|b|(6(q-p) + (2-p)(3-p)q)}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma+1)} \right. \\
& + \left. \left. \frac{|a+b|(6(q-p) + (2-p)(3-p)(6-q))\Gamma(4-\gamma)T^\alpha}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma+1)} \right) \right] \|u - v\|
\end{aligned}$$

for all $u, v \in X$ and $t \in I$. Similarly, one can obtain

$$\begin{aligned}
 & |(Tu)'(t) - (Tv)'(t)| \\
 & \leq \left[n_1 + n_2 + n_3 + n_4 + n_5 \gamma_0 \left(e_{11} + e_{21} + e_{31} + e_{41} \right. \right. \\
 & \quad \left. \left. + e_{51} \frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} + e_{61} \frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + e_{71} \frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)} \right) \right. \\
 & \quad \left. + n_6 \lambda_0 \left(e_{12} + e_{22} + e_{32} + e_{42} + e_{52} \frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} + e_{62} \frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + e_{72} \frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)} \right) \right. \\
 & \quad \left. + \sum_{i=1}^m k_i \frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} + \sum_{j=1}^{m'} k'_j \frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} + \sum_{k=1}^{m''} k''_k \frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)} \right] \\
 & \quad \times \left[\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{\Gamma(2-p)T^{\alpha-1}}{\Gamma(\alpha-p+1)} + \frac{(3-p)\Gamma(3-q)T^{\alpha-1}}{(2-p)\Gamma(\alpha-q+1)} \right. \\
 & \quad \left. + \frac{[2(q-p) + (2-p)(3-p)(5-q)]\Gamma(4-\gamma)T^{\alpha-1}}{2(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma+1)} \right] \|u-v\|, \\
 & |(Tu)''(t) - (Tv)''(t)| \\
 & \leq \left[n_1 + n_2 + n_3 + n_4 + n_5 \gamma_0 \left(e_{11} + e_{21} + e_{31} + e_{41} \right. \right. \\
 & \quad \left. \left. + e_{51} \frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} + e_{61} \frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + e_{71} \frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)} \right) \right. \\
 & \quad \left. + n_6 \lambda_0 \left(e_{12} + e_{22} + e_{32} + e_{42} + e_{52} \frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} + e_{62} \frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + e_{72} \frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)} \right) \right. \\
 & \quad \left. + \sum_{i=1}^m k_i \frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} + \sum_{j=1}^{m'} k'_j \frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} + \sum_{k=1}^{m''} k''_k \frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)} \right] \\
 & \quad \times \left[\frac{T^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{\Gamma(3-q)T^{\alpha-2}}{\Gamma(\alpha-q+1)} + \frac{(4-q)\Gamma(4-\gamma)T^{\alpha-2}}{(3-q)\Gamma(\alpha-\gamma+1)} \right] \|u-v\|,
 \end{aligned}$$

and

$$\begin{aligned}
 & |(Tu)'''(t) - (Tv)'''(t)| \\
 & \leq \left[n_1 + n_2 + n_3 + n_4 + n_5 \gamma_0 \left(e_{11} + e_{21} + e_{31} + e_{41} \right. \right. \\
 & \quad \left. \left. + e_{51} \frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} + e_{61} \frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + e_{71} \frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)} \right) \right. \\
 & \quad \left. + n_6 \lambda_0 \left(e_{12} + e_{22} + e_{32} + e_{42} + e_{52} \frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} + e_{62} \frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + e_{72} \frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)} \right) \right. \\
 & \quad \left. + \sum_{i=1}^m k_i \frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} + \sum_{j=1}^{m'} k'_j \frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} + \sum_{k=1}^{m''} k''_k \frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)} \right] \\
 & \quad \times \left[\frac{T^{\alpha-3}}{\Gamma(\alpha-2)} + \frac{\Gamma(4-\gamma)T^{\alpha-3}}{\Gamma(\alpha-\gamma+1)} \right]
 \end{aligned}$$

for all $u, v \in X$ and $t \in I$. This implies that

$$\begin{aligned} & \|Tu - Tv\| \\ & \leq M_1 \left[n_1 + n_2 + n_3 + n_4 + n_5 \gamma_0 \left(e_{11} + e_{21} + e_{31} + e_{41} \right. \right. \\ & \quad \left. \left. + e_{51} \frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} + e_{61} \frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + e_{71} \frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)} \right) \right. \\ & \quad \left. + n_6 \lambda_0 \left(e_{12} + e_{22} + e_{32} + e_{42} + e_{52} \frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} + e_{62} \frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + e_{72} \frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)} \right) \right. \\ & \quad \left. + \sum_{i=1}^m k_i \frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} + \sum_{j=1}^{m'} k'_j \frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} + \sum_{k=1}^{m''} k''_k \frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)} \right] \|u - v\| = \Delta \|u - v\|. \end{aligned}$$

Since $\Delta < 1$, the operator T is a contraction. In consequence, by the Banach contraction principle, T has a unique fixed point which corresponds to the unique solution of problem (1.1)-(1.3). \square

4 Existence results for problem (1.2)-(1.3)

This section is concerned with the existence of solutions for problem (1.2)-(1.3). As before, let the space $X = \{u : u \in C^3(I)\}$ be endowed with the norm

$$\|u\| = \sup_{t \in I} |u(t)| + \sup_{t \in I} |u'(t)| + \sup_{t \in I} |u''(t)| + \sup_{t \in I} |u'''(t)|.$$

A multivalued map $F : I \times \mathbb{R}^{6+m+m'+m''} \rightarrow P(\mathbb{R})$ is said to be Carathéodory whenever the map

$$t \mapsto F(t, x_1, x_2, \dots, x_{6+m+m'+m''})$$

is measurable for all $x_i \in \mathbb{R}$ for $1 \leq i \leq 6 + m + m' + m''$ and the map

$$(x_1, x_2, \dots, x_{6+m+m'+m''}) \mapsto F(t, x_1, x_2, \dots, x_{6+m+m'+m''})$$

is upper semi-continuous for almost all $t \in I$. Also, a Carathéodory function F is said to be L^1 -Carathéodory whenever for each $l > 0$ there exists $\psi_l \in L^1(I, \mathbb{R}^+)$ such that

$$\|F(t, x_1, x_2, \dots, x_{6+m+m'+m''})\|_p = \sup\{|v| : v \in F(t, x_1, x_2, \dots, x_{6+m+m'+m''})\} \leq \psi_l(t)$$

for all $|x_i| \leq l$ ($1 \leq i \leq 6 + m + m' + m''$) and almost all $t \in I$ [35, 37].

Lemma 4.1 ([41]) *Let E be a Banach space, $G : I \times E \rightarrow P_{cp,c}(E)$ an L^1 -Carathéodory multifunction, and θ a linear continuous mapping from $L^1(I, E)$ to $C(I, E)$. Then the operator*

$$\theta \circ S_G : C(I, E) \rightarrow P_{cp,c}(C(I, E))$$

defined by $\theta \circ S_G(x) = \theta(S_{G,x})$ is a closed graph operator, where

$$S_{G,x} = \{w \in L^1(I, E) : w(t) \in G(t, x(t)) \text{ for almost all } t \in I\}.$$

Let E be a Banach space. The multivalued map $G : I \times E \rightarrow P_{cp}(E)$ is said to be lower semi-continuous (l.s.c) type whenever $S_G : C(I, E) \rightarrow P(L^1(I, E))$ is lower semi-continuous and has non-empty closed and decomposable values [42].

Now we state some well-known results which are needed in the sequel.

Lemma 4.2 ([42]) *Let Y be a separable metric space and $N : Y \rightarrow P(L^1(I, \mathbb{R}))$ a lower semi-continuous multivalued map with closed decomposable values. Then N has a continuous selection, that is, there exists a continuous mapping $g : Y \rightarrow L^1(I, \mathbb{R})$ such that $g(y) \in N(y)$ for all $y \in Y$.*

Theorem 4.3 (Nonlinear alternative of Leray-Schauder type [43]) *Let E be a Banach space, C a closed and convex subset of E , and U an open subset of C such that $0 \in U$. If $F : \overline{U} \rightarrow P_{cp,c}(C)$ is an upper semi-continuous compact map, then either F has a fixed point in \overline{U} or there is a $x \in \partial U$ and $\lambda \in (0, 1)$ such that $x \in \lambda F(x)$.*

Theorem 4.4 (Covitz and Nadler [44]) *Let (M, d) be a complete metric space. If $F : M \rightarrow P_{cl}(M)$ is a contraction, then F has a fixed point.*

For $t \in I$ and $x \in X$, let the multifunction \tilde{F} be defined by

$$\begin{aligned} \tilde{F}(t, x(t)) = & F(t, x(t), x'(t), x''(t), x'''(t), \phi x(t), \psi x(t), {}^c D^{\mu_1} x(t), {}^c D^{\mu_2} x(t), \dots, {}^c D^{\mu_m} x(t), \\ & {}^c D^{\nu_1} x(t), {}^c D^{\nu_2} x(t), \dots, {}^c D^{\nu_{m'}} x(t), {}^c D^{\xi_1} x(t), {}^c D^{\xi_2} x(t), \dots, {}^c D^{\xi_{m''}} x(t)) \end{aligned}$$

and the set of selections of F by $S_{F,x} = \{v \in L^1(I, \mathbb{R}) : v(t) \in \tilde{F}(t, x(t)) \text{ for almost all } t \in I\}$. For the sake of brevity, we set

$$\begin{aligned} M_2 = & \left[\frac{|a| + 2|b|}{|a + b|\Gamma(\alpha)} + \frac{(|a| + 2|b|)\Gamma(2-p)}{|a + b|\Gamma(\alpha - p)} + \frac{(|b|p + |a + b|(4-p))\Gamma(3-q)}{2|a + b|(2-p)\Gamma(\alpha - q)} \right. \\ & + \left(\frac{[|b|(6(q-p) + (2-p)(3-p)q)]}{6|a + b|(2-p)(3-p)(3-q)\Gamma(\alpha - \gamma)} \right. \\ & \left. \left. + \frac{|a + b|(6(q-p) + (2-p)(3-p)(6-q))\Gamma(4-\gamma)}{6|a + b|(2-p)(3-p)(3-q)\Gamma(\alpha - \gamma)} \right) \right] T^{\alpha-1} \\ & + \left[\frac{1}{\Gamma(\alpha-1)} + \frac{\Gamma(2-p)}{\Gamma(\alpha-p)} + \frac{(3-p)\Gamma(3-q)}{(2-p)\Gamma(\alpha-q)} \right. \\ & \left. + \frac{[2(q-p) + (2-p)(3-p)(5-q)]\Gamma(4-\gamma)}{2(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)} \right] T^{\alpha-2} \\ & + \left[\frac{1}{\Gamma(\alpha-2)} + \frac{\Gamma(3-q)}{\Gamma(\alpha-q)} + \frac{(4-q)\Gamma(4-\gamma)}{(3-q)\Gamma(\alpha-\gamma)} \right] T^{\alpha-3} \\ & + \left[\frac{1}{\Gamma(\alpha-3)} + \frac{\Gamma(4-\gamma)}{\Gamma(\alpha-\gamma)} \right] T^{\alpha-4}. \end{aligned}$$

Now, we are in a position to give our first existence result for problem (1.2)-(1.3).

Theorem 4.5 *Suppose that $F : I \times \mathbb{R}^{6+m+m'+m''} \rightarrow P_{cp,c}(\mathbb{R})$ is a Carathéodory multivalued map and there exist continuous nondecreasing functions $\varphi_i : [0, \infty) \rightarrow (0, \infty)$ for $1 \leq i \leq 6$, $\psi_i, \psi'_j, \psi''_k : [0, \infty) \rightarrow (0, \infty)$ for $1 \leq i \leq m$, $1 \leq j \leq m'$, and $1 \leq k \leq m''$ and nonnegative*

functions $q_i \in L^1(I)$ for $1 \leq i \leq 6$, $\rho_i, \rho'_j, \rho''_k \in L^1(I)$ for $1 \leq i \leq m$, $1 \leq j \leq m'$ and $1 \leq k \leq m''$ such that

$$\begin{aligned} & \|F(t, x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2, \dots, y_m, z_1, z_2, \dots, z_{m'}, w_1, w_2, \dots, w_{m''})\|_p \\ &= \sup\{|y| : y \in F(t, x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2, \dots, y_m, z_1, z_2, \dots, z_{m'}, w_1, w_2, \dots, w_{m''})\} \\ &\leq q_1(t)\varphi_1(|x_1|) + q_2(t)\varphi_2(|x_2|) + q_3(t)\varphi_3(|x_3|) + q_4(t)\varphi_4(|x_4|) \\ &\quad + q_5(t)\varphi_5(|x_5|) + q_6(t)\varphi_6(|x_6|) \\ &\quad + \sum_{i=1}^m \rho_i(t)\psi_i(|y_i|) + \sum_{j=1}^{m'} \rho'_j(t)\psi'_j(|z_j|) + \sum_{k=1}^{m''} \rho''_k(t)\psi''_k(|w_k|), \end{aligned}$$

for all $t \in I$, $x_i, y_i, z_j, w_k \in \mathbb{R}$ for $1 \leq i \leq 6$, $1 \leq i \leq m$, $1 \leq j \leq m'$, and $1 \leq k \leq m''$. Assume that there exist positive constants $c_{01}, c_{02} > 0$ and $l_{i1}, l_{i2} \geq 0$ for $1 \leq i \leq 7$ such that

$$|h_j(t, s, u_1, u_2, u_3, u_4, u_5, u_6, u_7)| \leq c_{0j} + \sum_{i=1}^7 l_{ij}|u_i|$$

for $j = 1, 2$, all $t, s \in I$, and all $u_i \in \mathbb{R}$ for $1 \leq i \leq 7$. If there exists a constant $D > 0$ such that $\frac{D}{M_2 A(D)} > 1$, then problem (1.2)-(1.3) has at least one solution, where

$$\begin{aligned} A(D) &= \|q_1\|_1\varphi_1(D) + \|q_2\|_1\varphi_2(D) + \|q_3\|_1\varphi_3(D) + \|q_4\|_1\varphi_4(D) \\ &\quad + \|q_5\|_1\varphi_5\left(c_{01}\gamma_0 + \gamma_0\left[l_{11} + l_{21} + l_{31} + l_{41} + l_{51}\frac{T^{1-\gamma_1}}{\Gamma(2-\gamma_1)}\right.\right. \\ &\quad \left.\left.+ l_{61}\frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + l_{71}\frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)}\right]D\right) \\ &\quad + \|q_6\|_1\varphi_6\left(c_{02}\lambda_0 + \lambda_0\left[l_{12} + l_{22} + l_{32} + l_{42} + l_{52}\frac{T^{1-\gamma_2}}{\Gamma(2-\gamma_2)}\right.\right. \\ &\quad \left.\left.+ l_{62}\frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + l_{72}\frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)}\right]D\right) \\ &\quad + \sum_{i=1}^m \|\rho_i\|_1\psi_i\left(\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)}D\right) + \sum_{j=1}^{m'} \|\rho'_j\|_1\psi'_j\left(\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)}D\right) \\ &\quad + \sum_{k=1}^{m''} \|\rho''_k\|_1\psi''_k\left(\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)}D\right). \end{aligned}$$

Proof Let $x \in X$. Observe that the first property of the multifunction F and Theorem 1.3.5 in [45] imply that $S_{F,x}$ is non-empty. Define an operator $\Omega : X \rightarrow P(X)$ by

$$\Omega(x) = \{g \in X : \text{there exists } f \in S_{F,x} \text{ such that } g(t) = \gamma_f(t) \text{ for all } t \in I\},$$

where

$$\begin{aligned} \gamma_f(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s) ds \\ &\quad + \frac{[bT - (a+b)t]\Gamma(2-p)}{(a+b)\Gamma(\alpha-p)T^{1-p}} \int_0^T (T-s)^{\alpha-p-1} f(s) ds \end{aligned}$$

$$\begin{aligned} & - \frac{[bpT^2 - (a+b)(2Tt - (2-p)t^2)]\Gamma(3-q)}{2(a+b)(2-p)\Gamma(\alpha-q)T^{2-q}} \int_0^T (T-s)^{\alpha-q-1} f(s) ds \\ & - \left(\frac{[b(-6(q-p) + (2-p)(3-p)q)T^3]}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right. \\ & \left. + \frac{(a+b)(6(q-p)T^2t + (2-p)(3-p)(-3Tt^2 + (3-q)t^3))\Gamma(4-\gamma)}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right) \\ & \times \int_0^T (T-s)^{\alpha-\gamma-1} f(s) ds. \end{aligned}$$

We show that the operator Ω satisfies the hypothesis of the nonlinear alternative of the Leray-Schauder type result (Theorem 4.3). First, we show that $\Omega(x)$ is convex for all $x \in X$. Let $g_1, g_2 \in \Omega(x)$ and $w \in [0, 1]$. Choose $f_1, f_2 \in S_{F,x}$ such that

$$\begin{aligned} g_i(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_i(s) ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f_i(s) ds \\ & + \frac{[bT - (a+b)t]\Gamma(2-p)}{(a+b)\Gamma(\alpha-p)T^{1-p}} \int_0^T (T-s)^{\alpha-p-1} f_i(s) ds \\ & - \frac{[bpT^2 - (a+b)(2Tt - (2-p)t^2)]\Gamma(3-q)}{2(a+b)(2-p)\Gamma(\alpha-q)T^{2-q}} \int_0^T (T-s)^{\alpha-q-1} f_i(s) ds \\ & - \left(\frac{[b(-6(q-p) + (2-p)(3-p)q)T^3]}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right. \\ & \left. + \frac{(a+b)(6(q-p)T^2t + (2-p)(3-p)(-3Tt^2 + (3-q)t^3))\Gamma(4-\gamma)}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right) \\ & \times \int_0^T (T-s)^{\alpha-\gamma-1} f_i(s) ds \end{aligned}$$

for all $t \in I$. Then we have

$$\begin{aligned} & [wg_1 + (1-w)g_2](t) \\ & = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [wf_1(s) + (1-w)f_2(s)] ds \\ & - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} [wf_1(s) + (1-w)f_2(s)] ds \\ & + \frac{[bT - (a+b)t]\Gamma(2-p)}{(a+b)\Gamma(\alpha-p)T^{1-p}} \int_0^T (T-s)^{\alpha-p-1} [wf_1(s) + (1-w)f_2(s)] ds \\ & - \frac{[bpT^2 - (a+b)(2Tt - (2-p)t^2)]\Gamma(3-q)}{2(a+b)(2-p)\Gamma(\alpha-q)T^{2-q}} \\ & \times \int_0^T (T-s)^{\alpha-q-1} [wf_1(s) + (1-w)f_2(s)] ds \\ & - \left(\frac{[b(-6(q-p) + (2-p)(3-p)q)T^3]}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right. \\ & \left. + \frac{(a+b)(6(q-p)T^2t + (2-p)(3-p)(-3Tt^2 + (3-q)t^3))\Gamma(4-\gamma)}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right) \\ & \times \int_0^T (T-s)^{\alpha-\gamma-1} [wf_1(s) + (1-w)f_2(s)] ds \end{aligned}$$

for all $t \in I$. Since F has convex values, it is easy to check that $S_{F,x}$ is convex and so $wg_1 + (1-w)g_2 \in \Omega(x)$. Now, we show that Ω maps bounded sets into bounded sets in X . Let $B_r = \{x \in X : \|x\| \leq r\}$, $x \in B_r$, and $g \in \Omega(x)$. Choose $f \in S_{F,x}$ such that

$$\begin{aligned} |g(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s)| ds + \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |f(s)| ds \\ &\quad + \frac{|bT - (a+b)t|\Gamma(2-p)}{|a+b|\Gamma(\alpha-p)T^{1-p}} \int_0^T (T-s)^{\alpha-p-1} |f(s)| ds \\ &\quad + \frac{|bpT^2 - (a+b)(2Tt - (2-p)t^2)|\Gamma(3-q)}{2|a+b|(2-p)\Gamma(\alpha-q)T^{2-q}} \int_0^T (T-s)^{\alpha-q-1} |f(s)| ds \\ &\quad + \left(\frac{|b(-6(q-p) + (2-p)(3-p)q)T^3|}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right. \\ &\quad \left. + \frac{(a+b)(6(q-p)T^2t + (2-p)(3-p)(-3Tt^2 + (3-q)t^3))\Gamma(4-\gamma)}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right) \\ &\quad \times \int_0^T (T-s)^{\alpha-\gamma-1} |f(s)| ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[q_1(s)\varphi_1(|x(s)|) + q_2(s)\varphi_2(|x'(s)|) \right. \\ &\quad + q_3(s)\varphi_3(|x''(s)|) + q_4(s)\varphi_4(|x'''(s)|) \\ &\quad + q_5(s)\varphi_5(|\phi x(s)|) + q_6(s)\varphi_6(|\psi x(s)|) + \sum_{i=1}^m \rho_i(s)\psi_i(|{}^c D^{\mu_i} x(s)|) \\ &\quad \left. + \sum_{j=1}^{m'} \rho'_j(s)\psi'_j(|{}^c D^{\nu_j} x(s)|) + \sum_{k=1}^{m''} \rho''_k(s)\psi''_k(|{}^c D^{\xi_k} x(s)|) \right] ds \\ &\quad + \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \left[q_1(s)\varphi_1(|x(s)|) + q_2(s)\varphi_2(|x'(s)|) \right. \\ &\quad + q_3(s)\varphi_3(|x''(s)|) + q_4(s)\varphi_4(|x'''(s)|) + q_5(s)\varphi_5(|\phi x(s)|) \\ &\quad + q_6(s)\varphi_6(|\psi x(s)|) + \sum_{i=1}^m \rho_i(s)\psi_i(|{}^c D^{\mu_i} x(s)|) \\ &\quad \left. + \sum_{j=1}^{m'} \rho'_j(s)\psi'_j(|{}^c D^{\nu_j} x(s)|) + \sum_{k=1}^{m''} \rho''_k(s)\psi''_k(|{}^c D^{\xi_k} x(s)|) \right] ds \\ &\quad + \frac{|bT - (a+b)t|\Gamma(2-p)}{|a+b|\Gamma(\alpha-p)T^{1-p}} \\ &\quad \times \int_0^T (T-s)^{\alpha-p-1} \left[q_1(s)\varphi_1(|x(s)|) + q_2(s)\varphi_2(|x'(s)|) \right. \\ &\quad + q_3(s)\varphi_3(|x''(s)|) + q_4(s)\varphi_4(|x'''(s)|) \\ &\quad + q_5(s)\varphi_5(|\phi x(s)|) + q_6(s)\varphi_6(|\psi x(s)|) + \sum_{i=1}^m \rho_i(s)\psi_i(|{}^c D^{\mu_i} x(s)|) \\ &\quad \left. + \sum_{j=1}^{m'} \rho'_j(s)\psi'_j(|{}^c D^{\nu_j} x(s)|) + \sum_{k=1}^{m''} \rho''_k(s)\psi''_k(|{}^c D^{\xi_k} x(s)|) \right] ds \end{aligned}$$

$$\begin{aligned}
& + \frac{|bpT^2 - (a+b)(2Tt - (2-p)t^2)|\Gamma(3-q)}{2|a+b|(2-p)\Gamma(\alpha-q)T^{2-q}} \\
& \times \int_0^T (T-s)^{\alpha-q-1} \left[q_1(s)\varphi_1(|x(s)|) + q_2(s)\varphi_2(|x'(s)|) \right. \\
& + q_3(s)\varphi_3(|x''(s)|) + q_4(s)\varphi_4(|x'''(s)|) \\
& + q_5(s)\varphi_5(|\phi x(s)|) + q_6(s)\varphi_6(|\psi x(s)|) + \sum_{i=1}^m \rho_i(s)\psi_i(|{}^c D^{\mu_i} x(s)|) \\
& + \left. \sum_{j=1}^{m'} \rho'_j(s)\psi'_j(|{}^c D^{\nu_j} x(s)|) + \sum_{k=1}^{m''} \rho''_k(s)\psi''_k(|{}^c D^{\xi_k} x(s)|) \right] ds \\
& + \left(\frac{|b(-6(q-p) + (2-p)(3-p)q)T^3}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right. \\
& + \frac{(a+b)(6(q-p)T^2t + (2-p)(3-p)(-3Tt^2 + (3-q)t^3))|\Gamma(4-\gamma)}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \Big) \\
& \times \int_0^T (T-s)^{\alpha-\gamma-1} \left[q_1(s)\varphi_1(|x(s)|) + q_2(s)\varphi_2(|x'(s)|) \right. \\
& + q_3(s)\varphi_3(|x''(s)|) + q_4(s)\varphi_4(|x'''(s)|) \\
& + q_5(s)\varphi_5(|\phi x(s)|) + q_6(s)\varphi_6(|\psi x(s)|) + \sum_{i=1}^m \rho_i(s)\psi_i(|{}^c D^{\mu_i} x(s)|) \\
& + \left. \sum_{j=1}^{m'} \rho'_j(s)\psi'_j(|{}^c D^{\nu_j} x(s)|) + \sum_{k=1}^{m''} \rho''_k(s)\psi''_k(|{}^c D^{\xi_k} x(s)|) \right] ds \\
& \leq \left[\|q_1\|_1\varphi_1(r) + \|q_2\|_1\varphi_2(r) + \|q_3\|_1\varphi_3(r) + \|q_4\|_1\varphi_4(r) \right. \\
& + \|q_5\|_1\varphi_5 \left(c_{01}\gamma_0 + \gamma_0 \left[l_{11} + l_{21} + l_{31} + l_{41} + l_{51} \frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} \right. \right. \\
& + \left. \left. l_{61} \frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + l_{71} \frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)} \right] r \right) \\
& + \|q_6\|_1\varphi_6 \left(c_{02}\lambda_0 + \lambda_0 \left[l_{12} + l_{22} + l_{32} + l_{42} + l_{52} \frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} \right. \right. \\
& + \left. \left. l_{62} \frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + l_{72} \frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)} \right] r \right) \\
& + \sum_{i=1}^m \|\rho_i\|_1\psi_i \left(\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} r \right) + \sum_{j=1}^{m'} \|\rho'_j\|_1\psi'_j \left(\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} r \right) \\
& + \sum_{k=1}^{m''} \|\rho''_k\|_1\psi''_k \left(\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)} r \right) \Big] \\
& \times \left[\frac{(|a| + 2|b|)T^{\alpha-1}}{|a+b|\Gamma(\alpha)} + \frac{(|a| + 2|b|)\Gamma(2-p)T^{\alpha-1}}{|a+b|\Gamma(\alpha-p)} \right. \\
& + \left. \frac{(|b|p + |a+b|(4-p))\Gamma(3-q)T^{\alpha-1}}{2|a+b|(2-p)\Gamma(\alpha-q)} \right]
\end{aligned}$$

$$+ \left(\frac{[|b|(6(q-p) + (2-p)(3-p)q)]}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)} + \frac{|a+b|(6(q-p) + (2-p)(3-p)(6-q))\Gamma(4-\gamma)T^{\alpha-1}}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)} \right) \Bigg]$$

for all $t \in I$. In a similar manner, we obtain

$$\begin{aligned} |g'(t)| \leq & \left[\|q_1\|_1 \varphi_1(r) + \|q_2\|_1 \varphi_2(r) + \|q_3\|_1 \varphi_3(r) + \|q_4\|_1 \varphi_4(r) \right. \\ & + \|q_5\|_1 \varphi_5 \left(c_{01} \gamma_0 + \gamma_0 \left[l_{11} + l_{21} + l_{31} + l_{41} + l_{51} \frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} \right. \right. \\ & \left. \left. + l_{61} \frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + l_{71} \frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)} \right] r \right) \\ & + \|q_6\|_1 \varphi_6 \left(c_{02} \lambda_0 + \lambda_0 \left[l_{12} + l_{22} + l_{32} + l_{42} + l_{52} \frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} \right. \right. \\ & \left. \left. + l_{62} \frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + l_{72} \frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)} \right] r \right) \\ & + \sum_{i=1}^m \|\rho_i\|_1 \psi_i \left(\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} r \right) + \sum_{j=1}^{m'} \|\rho'_j\|_1 \psi'_j \left(\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} r \right) \\ & \left. + \sum_{k=1}^{m''} \|\rho''_k\|_1 \psi''_k \left(\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)} r \right) \right] \\ & \times \left[\frac{T^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{\Gamma(2-p)T^{\alpha-2}}{\Gamma(\alpha-p)} + \frac{(3-p)\Gamma(3-q)T^{\alpha-2}}{(2-p)\Gamma(\alpha-q)} \right. \\ & \left. + \frac{[2(q-p) + (2-p)(3-p)(5-q)]\Gamma(4-\gamma)T^{\alpha-2}}{2(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)} \right], \\ |g''(t)| \leq & \left[\|q_1\|_1 \varphi_1(r) + \|q_2\|_1 \varphi_2(r) + \|q_3\|_1 \varphi_3(r) + \|q_4\|_1 \varphi_4(r) \right. \\ & + \|q_5\|_1 \varphi_5 \left(c_{01} \gamma_0 + \gamma_0 \left[l_{11} + l_{21} + l_{31} + l_{41} + l_{51} \frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} \right. \right. \\ & \left. \left. + l_{61} \frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + l_{71} \frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)} \right] r \right) \\ & + \|q_6\|_1 \varphi_6 \left(c_{02} \lambda_0 + \lambda_0 \left[l_{12} + l_{22} + l_{32} + l_{42} + l_{52} \frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} \right. \right. \\ & \left. \left. + l_{62} \frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + l_{72} \frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)} \right] r \right) \\ & + \sum_{i=1}^m \|\rho_i\|_1 \psi_i \left(\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} r \right) + \sum_{j=1}^{m'} \|\rho'_j\|_1 \psi'_j \left(\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} r \right) \\ & \left. + \sum_{k=1}^{m''} \|\rho''_k\|_1 \psi''_k \left(\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)} r \right) \right] \\ & \times \left[\frac{T^{\alpha-3}}{\Gamma(\alpha-2)} + \frac{\Gamma(3-q)T^{\alpha-3}}{\Gamma(\alpha-q)} + \frac{(4-q)\Gamma(4-\gamma)T^{\alpha-3}}{(3-q)\Gamma(\alpha-\gamma)} \right], \end{aligned}$$

and

$$\begin{aligned}
 |g'''(t)| \leq & \left[\|q_1\|_1 \varphi_1(r) + \|q_2\|_1 \varphi_2(r) + \|q_3\|_1 \varphi_3(r) + \|q_4\|_1 \varphi_4(r) \right. \\
 & + \|q_5\|_1 \varphi_5 \left(c_{01} \gamma_0 + \gamma_0 \left[l_{11} + l_{21} + l_{31} + l_{41} + l_{51} \frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} \right. \right. \\
 & \left. \left. + l_{61} \frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + l_{71} \frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)} \right] r \right) \\
 & + \|q_6\|_1 \varphi_6 \left(c_{02} \lambda_0 + \lambda_0 \left[l_{12} + l_{22} + l_{32} + l_{42} + l_{52} \frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} \right. \right. \\
 & \left. \left. + l_{62} \frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + l_{72} \frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)} \right] r \right) \\
 & + \sum_{i=1}^m \|\rho_i\|_1 \psi_i \left(\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} r \right) + \sum_{j=1}^{m'} \|\rho'_j\|_1 \psi'_j \left(\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} r \right) \\
 & \left. + \sum_{k=1}^{m''} \|\rho''_k\|_1 \psi''_k \left(\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)} r \right) \right] \left[\frac{T^{\alpha-4}}{\Gamma(\alpha-3)} + \frac{\Gamma(4-\gamma) T^{\alpha-4}}{\Gamma(\alpha-\gamma)} \right]
 \end{aligned}$$

for all $t \in I$. Thus, we get

$$\begin{aligned}
 \|g\| \leq M_2 & \left[\|q_1\|_1 \varphi_1(r) + \|q_2\|_1 \varphi_2(r) + \|q_3\|_1 \varphi_3(r) + \|q_4\|_1 \varphi_4(r) \right. \\
 & + \|q_5\|_1 \varphi_5 \left(c_{01} \gamma_0 + \gamma_0 \left[l_{11} + l_{21} + l_{31} + l_{41} + l_{51} \frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} \right. \right. \\
 & \left. \left. + l_{61} \frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + l_{71} \frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)} \right] r \right) \\
 & + \|q_6\|_1 \varphi_6 \left(c_{02} \lambda_0 + \lambda_0 \left[l_{12} + l_{22} + l_{32} + l_{42} + l_{52} \frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} \right. \right. \\
 & \left. \left. + l_{62} \frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + l_{72} \frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)} \right] r \right) \\
 & + \sum_{i=1}^m \|\rho_i\|_1 \psi_i \left(\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} r \right) + \sum_{j=1}^{m'} \|\rho'_j\|_1 \psi'_j \left(\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} r \right) \\
 & \left. + \sum_{k=1}^{m''} \|\rho''_k\|_1 \psi''_k \left(\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)} r \right) \right].
 \end{aligned}$$

This implies that Ω maps bounded sets into bounded sets in X . Now, we show that Ω maps bounded sets of X into equicontinuous sets. Let $t_1, t_2 \in [0, T]$ with $t_1 < t_2$, $x \in B_r$, and $g \in \Omega(x)$. Then we have

$$\begin{aligned}
 & |g(t_2) - g(t_1)| \\
 & \leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] |f(s)| ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} |f(s)| ds
 \end{aligned}$$

$$\begin{aligned}
& + \frac{(t_2 - t_1)\Gamma(2-p)}{\Gamma(\alpha-p)T^{1-p}} \int_0^T (T-s)^{\alpha-p-1} |f(s)| ds \\
& + \frac{[2T(t_2 - t_1) + (2-p)(t_2^2 - t_1^2)]\Gamma(3-q)}{2(2-p)\Gamma(\alpha-q)T^{2-q}} \int_0^T (T-s)^{\alpha-q-1} |f(s)| ds \\
& + \frac{[6(q-p)T^2(t_2 - t_1) + (2-p)(3-p)(3T(t_2^2 - t_1^2) + (3-q)(t_2^3 - t_1^3))]\Gamma(4-\gamma)}{6(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \\
& \times \int_0^T (T-s)^{\alpha-\gamma-1} |f(s)| ds \\
\leq & \left[\varphi_1(r) + \varphi_2(r) + \varphi_3(r) + \varphi_4(r) \right. \\
& + \varphi_5 \left(c_{01}\gamma_0 + \gamma_0 \left[l_{11} + l_{21} + l_{31} + l_{41} + l_{51} \frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} \right. \right. \\
& \left. \left. + l_{61} \frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + l_{71} \frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)} \right] r \right) \\
& + \varphi_6 \left(c_{02}\lambda_0 + \lambda_0 \left[l_{12} + l_{22} + l_{32} + l_{42} + l_{52} \frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} \right. \right. \\
& \left. \left. + l_{62} \frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + l_{72} \frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)} \right] r \right) \\
& + \sum_{i=1}^m \psi_i \left(\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} r \right) + \sum_{j=1}^{m'} \psi'_j \left(\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} r \right) + \sum_{k=1}^{m''} \psi''_k \left(\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)} r \right) \Big] \\
& \times \left[\frac{1}{\Gamma(\alpha)} \int_0^{t_1} [(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}] \right. \\
& \times \left[\sum_{l=1}^6 q_l(s) + \sum_{i=1}^m \rho_i(s) + \sum_{j=1}^{m'} \rho'_j(s) + \sum_{k=1}^{m''} \rho''_k(s) \right] ds \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} \left[\sum_{l=1}^6 q_l(s) + \sum_{i=1}^m \rho_i(s) + \sum_{j=1}^{m'} \rho'_j(s) + \sum_{k=1}^{m''} \rho''_k(s) \right] ds \\
& + \frac{(t_2 - t_1)\Gamma(2-p)}{\Gamma(\alpha-p)T^{1-p}} \int_0^T (T-s)^{\alpha-p-1} \\
& \times \left[\sum_{l=1}^6 q_l(s) + \sum_{i=1}^m \rho_i(s) + \sum_{j=1}^{m'} \rho'_j(s) + \sum_{k=1}^{m''} \rho''_k(s) \right] ds \\
& + \frac{[2T(t_2 - t_1) + (2-p)(t_2^2 - t_1^2)]\Gamma(3-q)}{2(2-p)\Gamma(\alpha-q)T^{2-q}} \\
& \times \int_0^T (T-s)^{\alpha-q-1} \left[\sum_{l=1}^6 q_l(s) + \sum_{i=1}^m \rho_i(s) + \sum_{j=1}^{m'} \rho'_j(s) + \sum_{k=1}^{m''} \rho''_k(s) \right] ds \\
& + \frac{[6(q-p)T^2(t_2 - t_1) + (2-p)(3-p)(3T(t_2^2 - t_1^2) + (3-q)(t_2^3 - t_1^3))]\Gamma(4-\gamma)}{6(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \\
& \times \int_0^T (T-s)^{\alpha-\gamma-1} \left[\sum_{l=1}^6 q_l(s) + \sum_{i=1}^m \rho_i(s) + \sum_{j=1}^{m'} \rho'_j(s) + \sum_{k=1}^{m''} \rho''_k(s) \right] ds \Big].
\end{aligned}$$

Proceeding as before, one can obtain

$$\begin{aligned}
 & |g'(t_2) - g'(t_1)| \\
 & \leq \left[\varphi_1(r) + \varphi_2(r) + \varphi_3(r) + \varphi_4(r) \right. \\
 & \quad + \varphi_5 \left(c_{01}\gamma_0 + \gamma_0 \left[l_{11} + l_{21} + l_{31} + l_{41} + l_{51} \frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} \right. \right. \\
 & \quad \left. \left. + l_{61} \frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + l_{71} \frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)} \right] r \right) \\
 & \quad + \varphi_6 \left(c_{02}\lambda_0 + \lambda_0 \left[l_{12} + l_{22} + l_{32} + l_{42} + l_{52} \frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} \right. \right. \\
 & \quad \left. \left. + l_{62} \frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + l_{72} \frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)} \right] r \right) \\
 & \quad + \sum_{i=1}^m \psi_i \left(\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} r \right) + \sum_{j=1}^{m'} \psi'_j \left(\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} r \right) + \sum_{k=1}^{m''} \psi''_k \left(\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)} r \right) \Big] \\
 & \quad \times \left[\frac{1}{\Gamma(\alpha-1)} \int_0^{t_1} [(t_2-s)^{\alpha-2} - (t_1-s)^{\alpha-2}] \right. \\
 & \quad \times \left[\sum_{l=1}^6 q_l(s) + \sum_{i=1}^m \rho_i(s) + \sum_{j=1}^{m'} \rho'_j(s) + \sum_{k=1}^{m''} \rho''_k(s) \right] ds \\
 & \quad + \frac{1}{\Gamma(\alpha-1)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-2} \left[\sum_{l=1}^6 q_l(s) + \sum_{i=1}^m \rho_i(s) + \sum_{j=1}^{m'} \rho'_j(s) + \sum_{k=1}^{m''} \rho''_k(s) \right] ds \\
 & \quad + \frac{(t_2-t_1)\Gamma(3-q)}{\Gamma(\alpha-q)T^{2-q}} \int_0^T (T-s)^{\alpha-q-1} \\
 & \quad \times \left[\sum_{l=1}^6 q_l(s) + \sum_{i=1}^m \rho_i(s) + \sum_{j=1}^{m'} \rho'_j(s) + \sum_{k=1}^{m''} \rho''_k(s) \right] ds \\
 & \quad + \frac{[2T(t_2-t_1) + (3-q)(t_2^2-t_1^2)]\Gamma(4-\gamma)}{2(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \\
 & \quad \times \int_0^T (T-s)^{\alpha-\gamma-1} \left[\sum_{l=1}^6 q_l(s) + \sum_{i=1}^m \rho_i(s) + \sum_{j=1}^{m'} \rho'_j(s) + \sum_{k=1}^{m''} \rho''_k(s) \right] ds \Big], \\
 & |g''(t_2) - g''(t_1)| \\
 & \leq \left[\varphi_1(r) + \varphi_2(r) + \varphi_3(r) + \varphi_4(r) + \varphi_5 \left(c_{01}\gamma_0 + \gamma_0 \left[l_{11} + l_{21} + l_{31} + l_{41} + l_{51} \frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} \right. \right. \right. \\
 & \quad \left. \left. + l_{61} \frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + l_{71} \frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)} \right] r \right) \\
 & \quad + \varphi_6 \left(c_{02}\lambda_0 + \lambda_0 \left[l_{12} + l_{22} + l_{32} + l_{42} + l_{52} \frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} \right. \right. \\
 & \quad \left. \left. + l_{62} \frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + l_{72} \frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)} \right] r \right) \\
 & \quad \left. + l_{61} \frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + l_{71} \frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)} \right] r \Big]
 \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^m \psi_i \left(\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} r \right) + \sum_{j=1}^{m'} \psi_j' \left(\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} r \right) + \sum_{k=1}^{m''} \psi_k'' \left(\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)} r \right) \\
& \times \left[\frac{1}{\Gamma(\alpha-2)} \int_0^{t_1} [(t_2-s)^{\alpha-3} - (t_1-s)^{\alpha-3}] \right. \\
& \times \left[\sum_{l=1}^6 q_l(s) + \sum_{i=1}^m \rho_i(s) + \sum_{j=1}^{m'} \rho_j'(s) + \sum_{k=1}^{m''} \rho_k''(s) \right] ds \\
& + \frac{1}{\Gamma(\alpha-2)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-3} \left[\sum_{l=1}^6 q_l(s) + \sum_{i=1}^m \rho_i(s) + \sum_{j=1}^{m'} \rho_j'(s) + \sum_{k=1}^{m''} \rho_k''(s) \right] ds \\
& + \frac{(t_2-t_1)\Gamma(4-\gamma)}{\Gamma(\alpha-\gamma)T^{3-\gamma}} \int_0^T (T-s)^{\alpha-\gamma-1} \\
& \times \left[\sum_{l=1}^6 q_l(s) + \sum_{i=1}^m \rho_i(s) + \sum_{j=1}^{m'} \rho_j'(s) + \sum_{k=1}^{m''} \rho_k''(s) \right] ds \Big],
\end{aligned}$$

and

$$\begin{aligned}
& |g'''(t_2) - g'''(t_1)| \\
& \leq \left[\varphi_1(r) + \varphi_2(r) + \varphi_3(r) + \varphi_4(r) + \varphi_5 \left(c_{01}\gamma_0 + \gamma_0 \left[l_{11} + l_{21} + l_{31} + l_{41} + l_{51} \frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} \right. \right. \right. \\
& \quad \left. \left. + l_{61} \frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + l_{71} \frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)} \right] r \right) \\
& \quad + \varphi_6 \left(c_{02}\lambda_0 + \lambda_0 \left[l_{12} + l_{22} + l_{32} + l_{42} + l_{52} \frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} \right. \right. \\
& \quad \left. \left. + l_{62} \frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + l_{72} \frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)} \right] r \right) \\
& \quad + \sum_{i=1}^m \psi_i \left(\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} r \right) + \sum_{j=1}^{m'} \psi_j' \left(\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} r \right) + \sum_{k=1}^{m''} \psi_k'' \left(\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)} r \right) \\
& \quad \times \left[\frac{1}{\Gamma(\alpha-3)} \int_0^{t_1} [(t_2-s)^{\alpha-4} - (t_1-s)^{\alpha-4}] \right. \\
& \quad \times \left[\sum_{l=1}^6 q_l(s) + \sum_{i=1}^m \rho_i(s) + \sum_{j=1}^{m'} \rho_j'(s) + \sum_{k=1}^{m''} \rho_k''(s) \right] ds \\
& \quad \left. + \frac{1}{\Gamma(\alpha-3)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-4} \left[\sum_{l=1}^6 q_l(s) + \sum_{i=1}^m \rho_i(s) + \sum_{j=1}^{m'} \rho_j'(s) + \sum_{k=1}^{m''} \rho_k''(s) \right] ds \right].
\end{aligned}$$

It is easy to see that the right-hand side of the above inequalities tends to zero as $t_2 - t_1 \rightarrow 0$ (independent on $x \in B_r$). Thus, by using the Arzela-Ascoli theorem, we see that $\Omega : X \rightarrow P(X)$ is a compact multivalued map. Next, we show that Ω has a closed graph. Let $x_n \rightarrow x_*$, $g_n \in \Omega(x_n)$ for all n and $g_n \rightarrow g_*$. We show that $g_* \in \Omega(x_*)$. Since $g_n \in \Omega(x_n)$ for all n , there

exists $f_n \in S_{F,x_n}$ such that

$$\begin{aligned} g_n(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_n(s) ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f_n(s) ds \\ & + \frac{[bT - (a+b)t]\Gamma(2-p)}{(a+b)\Gamma(\alpha-p)T^{1-p}} \int_0^T (T-s)^{\alpha-p-1} f_n(s) ds \\ & - \frac{[bpT^2 - (a+b)(2Tt - (2-p)t^2)]\Gamma(3-q)}{2(a+b)(2-p)\Gamma(\alpha-q)T^{2-q}} \int_0^T (T-s)^{\alpha-q-1} f_n(s) ds \\ & - \left(\frac{[b(-6(q-p) + (2-p)(3-p)q)T^3]}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right. \\ & \left. + \frac{(a+b)(6(q-p)T^2t + (2-p)(3-p)(-3Tt^2 + (3-q)t^3))\Gamma(4-\gamma)}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right) \\ & \times \int_0^T (T-s)^{\alpha-\gamma-1} f_n(s) ds \end{aligned}$$

for all $t \in I$. Thus, we have to show that there exists $f_* \in S_{F,x_*}$ such that

$$\begin{aligned} g_*(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_*(s) ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f_*(s) ds \\ & + \frac{[bT - (a+b)t]\Gamma(2-p)}{(a+b)\Gamma(\alpha-p)T^{1-p}} \int_0^T (T-s)^{\alpha-p-1} f_*(s) ds \\ & - \frac{[bpT^2 - (a+b)(2Tt - (2-p)t^2)]\Gamma(3-q)}{2(a+b)(2-p)\Gamma(\alpha-q)T^{2-q}} \int_0^T (T-s)^{\alpha-q-1} f_*(s) ds \\ & - \left(\frac{[b(-6(q-p) + (2-p)(3-p)q)T^3]}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right. \\ & \left. + \frac{(a+b)(6(q-p)T^2t + (2-p)(3-p)(-3Tt^2 + (3-q)t^3))\Gamma(4-\gamma)}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right) \\ & \times \int_0^T (T-s)^{\alpha-\gamma-1} f_*(s) ds \end{aligned}$$

for all $t \in I$. Consider the linear continuous operator $\theta : L^1(I, \mathbb{R}) \rightarrow X$ defined by $f \mapsto \theta(f)(t)$, where

$$\begin{aligned} \theta(f)(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s) ds \\ & + \frac{[bT - (a+b)t]\Gamma(2-p)}{(a+b)\Gamma(\alpha-p)T^{1-p}} \int_0^T (T-s)^{\alpha-p-1} f(s) ds \\ & - \frac{[bpT^2 - (a+b)(2Tt - (2-p)t^2)]\Gamma(3-q)}{2(a+b)(2-p)\Gamma(\alpha-q)T^{2-q}} \int_0^T (T-s)^{\alpha-q-1} f(s) ds \\ & - \left(\frac{[b(-6(q-p) + (2-p)(3-p)q)T^3]}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right. \\ & \left. + \frac{(a+b)(6(q-p)T^2t + (2-p)(3-p)(-3Tt^2 + (3-q)t^3))\Gamma(4-\gamma)}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right) \\ & \times \int_0^T (T-s)^{\alpha-\gamma-1} f(s) ds \end{aligned}$$

for all $t \in I$. Since θ is a linear continuous map, therefore, by Lemma 4.1, it follows that $\theta \circ S_F$ is a closed graph operator. Note that $g_n \in \theta \circ S_F(x_n)$ for all n . Since $x_n \rightarrow x_*$ and $g_n \rightarrow g_*$, there exists $f_* \in S_F(x_*)$ such that

$$\begin{aligned} g_*(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_*(s) ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f_*(s) ds \\ & + \frac{[bT - (a+b)t]\Gamma(2-p)}{(a+b)\Gamma(\alpha-p)T^{1-p}} \int_0^T (T-s)^{\alpha-p-1} f_*(s) ds \\ & - \frac{[bpT^2 - (a+b)(2Tt - (2-p)t^2)]\Gamma(3-q)}{2(a+b)(2-p)\Gamma(\alpha-q)T^{2-q}} \int_0^T (T-s)^{\alpha-q-1} f_*(s) ds \\ & - \left(\frac{[b(-6(q-p) + (2-p)(3-p)q)T^3]}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right. \\ & \left. + \frac{(a+b)(6(q-p)T^2t + (2-p)(3-p)(-3Tt^2 + (3-q)t^3))\Gamma(4-\gamma)}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right) \\ & \times \int_0^T (T-s)^{\alpha-\gamma-1} f_*(s) ds \end{aligned}$$

for all $t \in I$. Let $\lambda \in (0, 1)$ and $x \in \lambda\Omega(x)$, then there exists $f \in S_{F,x}$ such that

$$\begin{aligned} x(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t \lambda(t-s)^{\alpha-1} f(s) ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T \lambda(T-s)^{\alpha-1} f(s) ds \\ & + \frac{[bT - (a+b)t]\Gamma(2-p)}{(a+b)\Gamma(\alpha-p)T^{1-p}} \int_0^T \lambda(T-s)^{\alpha-p-1} f(s) ds \\ & - \frac{[bpT^2 - (a+b)(2Tt - (2-p)t^2)]\Gamma(3-q)}{2(a+b)(2-p)\Gamma(\alpha-q)T^{2-q}} \int_0^T \lambda(T-s)^{\alpha-q-1} f(s) ds \\ & - \left(\frac{[b(-6(q-p) + (2-p)(3-p)q)T^3]}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right. \\ & \left. + \frac{(a+b)(6(q-p)T^2t + (2-p)(3-p)(-3Tt^2 + (3-q)t^3))\Gamma(4-\gamma)}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right) \\ & \times \int_0^T \lambda(T-s)^{\alpha-\gamma-1} f(s) ds \end{aligned}$$

for all $t \in I$. Hence

$$\begin{aligned} \|x\| = & \sup_{t \in I} |x(t)| + \sup_{t \in I} |x'(t)| + \sup_{t \in I} |x''(t)| + \sup_{t \in I} |x'''(t)| \\ \leq & M_2 \left[\|q_1\|_1 \varphi_1(\|x\|) + \|q_2\|_1 \varphi_2(\|x\|) + \|q_3\|_1 \varphi_3(\|x\|) + \|q_4\|_1 \varphi_4(\|x\|) \right. \\ & + \|q_5\|_1 \varphi_5 \left(c_{01} \gamma_0 + \gamma_0 \left[l_{11} + l_{21} + l_{31} + l_{41} + l_{51} \frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} \right. \right. \\ & \left. \left. + l_{61} \frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + l_{71} \frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)} \right] \|x\| \right) \\ & \left. + \|q_6\|_1 \varphi_6 \left(c_{02} \lambda_0 + \lambda_0 \left[l_{12} + l_{22} + l_{32} + l_{42} + l_{52} \frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} \right. \right. \right. \end{aligned}$$

$$\begin{aligned}
& + l_{62} \frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + l_{72} \frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)} \Big] \|x\| \Big) \\
& + \sum_{i=1}^m \|\rho_i\|_1 \psi_i \left(\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} \|x\| \right) + \sum_{j=1}^{m'} \|\rho'_j\|_1 \psi'_j \left(\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} \|x\| \right) \\
& + \sum_{k=1}^{m''} \|\rho''_k\|_1 \psi''_k \left(\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)} \|x\| \right) \Big] \\
& = M_2 A(\|x\|)
\end{aligned}$$

and so $\frac{\|x\|}{M_2 A(\|x\|)} \leq 1$. Letting $U = \{x \in X : \|x\| < D\}$, the operator $\Omega : \overline{U} \rightarrow P_{cp,c}(X)$ is upper semi-continuous and compact. In view of the choice of U , there is no $x \in \partial U$ such that $x \in \lambda \Omega(x)$ for some $\lambda \in (0, 1)$ and so Ω has a fixed point $x \in \overline{U}$ by virtue of Theorem 4.3. Obviously, each fixed point of Ω is a solution of problem (1.2)-(1.3). This completes the proof. \square

Our next result deals with the case that F is not necessary convex valued.

Theorem 4.6 *Suppose that $F : I \times \mathbb{R}^{6+m+m'+m''} \rightarrow P_{cp}(\mathbb{R})$ is a multifunction such that the map*

$$(t, x_1, x_2, \dots, x_{6+m+m'+m''}) \mapsto F(t, x_1, x_2, \dots, x_{6+m+m'+m''})$$

is $L(I) \otimes B(\mathbb{R}) \otimes \dots \otimes B(\mathbb{R}) \otimes B(\mathbb{R})$ measurable and the map

$$(x_1, x_2, \dots, x_{6+m+m'+m''}) \mapsto F(t, x_1, x_2, \dots, x_{6+m+m'+m''})$$

is lower semi-continuous for almost all $t \in I$. Assume that there exist continuous non-decreasing functions $\varphi_i : [0, \infty) \rightarrow (0, \infty)$ for $1 \leq i \leq 6$, $\psi_i, \psi'_j, \psi''_k : [0, \infty) \rightarrow (0, \infty)$ for $1 \leq i \leq m, 1 \leq j \leq m',$ and $1 \leq k \leq m''$ and nonnegative functions $q_i \in L^1(I)$ for $1 \leq i \leq 6$, $\rho_i, \rho'_j, \rho''_k \in L^1(I)$ for $1 \leq i \leq m, 1 \leq j \leq m',$ and $1 \leq k \leq m''$ such that

$$\begin{aligned}
& \|F(t, x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2, \dots, y_m, z_1, z_2, \dots, z_{m'}, w_1, w_2, \dots, w_{m''})\|_p \\
& = \sup \{ |y| : y \in F(t, x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2, \dots, y_m, z_1, z_2, \dots, z_{m'}, w_1, w_2, \dots, w_{m''}) \} \\
& \leq q_1(t) \varphi_1(|x_1|) + q_2(t) \varphi_2(|x_2|) + q_3(t) \varphi_3(|x_3|) \\
& \quad + q_4(t) \varphi_4(|x_4|) + q_5(t) \varphi_5(|x_5|) + q_6(t) \varphi_6(|x_6|) \\
& \quad + \sum_{i=1}^m \rho_i(t) \psi_i(|y_i|) + \sum_{j=1}^{m'} \rho'_j(t) \psi'_j(|z_j|) + \sum_{k=1}^{m''} \rho''_k(t) \psi''_k(|w_k|),
\end{aligned}$$

for all $t \in I$, $x_i, y_i, z_j, w_k \in \mathbb{R}$ for $1 \leq i \leq 6, 1 \leq i \leq m, 1 \leq j \leq m',$ and $1 \leq k \leq m''$. Furthermore, there exist positive constants $c_{01}, c_{02} > 0$, and $l_{i1}, l_{i2} \geq 0$ for $1 \leq i \leq 7$ such that

$$|h_j(t, s, u_1, u_2, u_3, u_4, u_5, u_6, u_7)| \leq c_{0j} + \sum_{i=1}^7 l_{ij} |u_i|$$

for $j = 1, 2, t, s \in I$ and all $u_i \in \mathbb{R}$ for $1 \leq i \leq 7$. If there exists a constant $D > 0$ such that $\frac{D}{M_2 A(D)} > 1$, then problem (1.2)-(1.3) has at least one solution, where

$$\begin{aligned} A(D) = & \|q_1\|_1 \varphi_1(D) + \|q_2\|_1 \varphi_2(D) + \|q_3\|_1 \varphi_3(D) + \|q_4\|_1 \varphi_4(D) \\ & + \|q_5\|_1 \varphi_5 \left(c_{01} \gamma_0 + \gamma_0 \left[l_{11} + l_{21} + l_{31} + l_{41} + l_{51} \frac{T^{1-\gamma_1}}{\Gamma(2-\gamma_1)} \right. \right. \\ & \left. \left. + l_{61} \frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + l_{71} \frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)} \right] D \right) \\ & + \|q_6\|_1 \varphi_6 \left(c_{02} \lambda_0 + \lambda_0 \left[l_{12} + l_{22} + l_{32} + l_{42} + l_{52} \frac{T^{1-\gamma_2}}{\Gamma(2-\gamma_2)} \right. \right. \\ & \left. \left. + l_{62} \frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + l_{72} \frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)} \right] D \right) \\ & + \sum_{i=1}^m \|\rho_i\|_1 \psi_i \left(\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} D \right) + \sum_{j=1}^{m'} \|\rho'_j\|_1 \psi'_j \left(\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} D \right) \\ & + \sum_{k=1}^{m''} \|\rho''_k\|_1 \psi''_k \left(\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)} D \right). \end{aligned}$$

Proof In view of the given assumptions and Lemma 4.1 in [46], it follows that F is of lower semi-continuous type. Thus, by Lemma 4.2, there exists a continuous function $f : X \rightarrow L^1(I, \mathbb{R})$ such that $f(x) \in S_F(x)$ for all $x \in X$. Now, consider the equation

$${}^c D^\alpha x(t) = f(x)(t) \quad (4.1)$$

supplemented with boundary conditions (1.3). Note that each solution of problem (4.1)-(1.3) with the given conditions is a solution of problem (1.2)-(1.3). Define the operator $\overline{\Omega} : X \rightarrow X$ by

$$\begin{aligned} \overline{\Omega} x(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(x)(s) ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(x)(s) ds \\ & + \frac{[bT - (a+b)t]\Gamma(2-p)}{(a+b)\Gamma(\alpha-p)T^{1-p}} \int_0^T (T-s)^{\alpha-p-1} f(x)(s) ds \\ & - \frac{[bpT^2 - (a+b)(2Tt - (2-p)t^2)]\Gamma(3-q)}{2(a+b)(2-p)\Gamma(\alpha-q)T^{2-q}} \int_0^T (T-s)^{\alpha-q-1} f(x)(s) ds \\ & - \left(\frac{[b(-6(q-p) + (2-p)(3-p)q)T^3]}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right. \\ & \left. + \frac{(a+b)(6(q-p)T^2t + (2-p)(3-p)(-3Tt^2 + (3-q)t^3))\Gamma(4-\gamma)}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right) \\ & \times \int_0^T (T-s)^{\alpha-\gamma-1} f(x)(s) ds \end{aligned}$$

for all $t \in I$. Following the procedure employed in the last result, one can show that $\overline{\Omega}$ is continuous and completely continuous and satisfies all conditions of the nonlinear alternative of Leary-Schauder type for single-valued maps. Consequently, there exists a solution for problem (4.1)-(1.3). This completes the proof. \square

Finally, we establish the existence of a solution for the case that the right-hand side of (1.2) is non-convex valued.

Theorem 4.7 Assume that $F : I \times \mathbb{R}^{6+m+m'} \rightarrow P_{cp}(\mathbb{R})$ is a multifunction such that the map

$$t \mapsto F(t, x_1, x_2, \dots, x_{6+m+m'})$$

is measurable for all $x_1, \dots, x_{6+m+m'} \in \mathbb{R}$, the map $t \mapsto d_H(0, F(t, 0, 0, \dots, 0))$ is integrably bounded for almost all $t \in I$ and there are nonnegative functions $k_1, \dots, k_6 \in L^1(I)$, $m_1, \dots, m_m, m'_1, \dots, m'_{m'}, m''_1, \dots, m''_{m''} \in L^1(I)$ such that

$$\begin{aligned} & d_H(F(t, x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2, \dots, y_m, z_1, z_2, \dots, z_{m'}, w_1, w_2, \dots, w_{m''}), \\ & F(t, x'_1, x'_2, x'_3, x'_4, x'_5, x'_6, y'_1, y'_2, \dots, y'_m, z'_1, z'_2, \dots, z'_{m'}, w'_1, w'_2, \dots, w'_{m''})) \\ & \leq k_1(t)|x_1 - x'_1| + k_2(t)|x_2 - x'_2| + k_3(t)|x_3 - x'_3| \\ & \quad + k_4(t)|x_4 - x'_4| + k_5(t)|x_5 - x'_5| + k_6(t)|x_6 - x'_6| \\ & \quad + \sum_{i=1}^m m_i(t)|y_i - y'_i| + \sum_{j=1}^{m'} m'_j(t)|z_j - z'_j| + \sum_{k=1}^{m''} m''_k(t)|w_k - w'_k| \end{aligned}$$

for almost all $t \in I$ and all $x_1, \dots, x_6, x'_1, \dots, x'_6, y_1, \dots, y_m, y'_1, \dots, y'_m, z_1, \dots, z_{m'}, z'_1, \dots, z'_{m'}, w_1, \dots, w_{m''}, w'_1, \dots, w'_{m''} \in \mathbb{R}$. Also, suppose that there exist $\eta_{11}, \dots, \eta_{71}, \eta_{12}, \dots, \eta_{72} \geq 0$ such that

$$|h_j(t, s, u_1, u_2, u_3, u_4, u_5, u_6, u_7) - h_j(t, s, u'_1, u'_2, u'_3, u'_4, u'_5, u'_6, u'_7)| \leq \sum_{i=1}^7 \eta_{ij} |u_i - u'_i|$$

for all $t, s \in I$, $u_1, \dots, u_7, u'_1, \dots, u'_7 \in \mathbb{R}$, and $j = 1, 2$. If $\Theta < 1$, then problem (1.2)-(1.3) has at least one solution, where

$$\begin{aligned} \Theta = M_2 & \left[\|k_1\|_1 + \|k_2\|_1 + \|k_3\|_1 + \|k_4\|_1 \right. \\ & + \|k_5\|_1 \gamma_0 \left(\eta_{11} + \eta_{21} + \eta_{31} + \eta_{41} + \eta_{51} \frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} + \eta_{61} \frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + \eta_{71} \frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)} \right) \\ & + \|k_6\|_1 \lambda_0 \left(\eta_{12} + \eta_{22} + \eta_{32} + \eta_{42} + \eta_{52} \frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} + \eta_{62} \frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + \eta_{72} \frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)} \right) \\ & \left. + \sum_{i=1}^m \|m_i\|_1 \frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} + \sum_{j=1}^{m'} \|m'_j\|_1 \frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} + \sum_{k=1}^{m''} \|m''_k\|_1 \frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)} \right]. \end{aligned}$$

Proof With the given assumptions and Theorem III-6 (the measurable selection theorem) in [47], one can infer that F admits a measurable selection $f : I \rightarrow \mathbb{R}$. Since F is integrably bounded, $f \in L^1(I, \mathbb{R})$, so $S_{F,x} \neq \emptyset$ for all $x \in X$. Now, we show that the operator Ω satisfies the assumptions of Theorem 4.4. First, we show that $\Omega(x) \in P_{cl}(X)$ for all $x \in X$. Let $u_n \in$

$\Omega(x)$ for all $n \geq 0$ and $u_n \rightarrow u$ for some $u \in X$. For each n , choose $v_n \in S_{F,x}$ such that

$$\begin{aligned} u_n(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_n(s) ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} v_n(s) ds \\ & + \frac{[bT - (a+b)t]\Gamma(2-p)}{(a+b)\Gamma(\alpha-p)T^{1-p}} \int_0^T (T-s)^{\alpha-p-1} v_n(s) ds \\ & - \frac{[bpT^2 - (a+b)(2Tt - (2-p)t^2)]\Gamma(3-q)}{2(a+b)(2-p)\Gamma(\alpha-q)T^{2-q}} \int_0^T (T-s)^{\alpha-q-1} v_n(s) ds \\ & - \left(\frac{[b(-6(q-p) + (2-p)(3-p)q)T^3]}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right. \\ & \left. + \frac{(a+b)(6(q-p)T^2t + (2-p)(3-p)(-3Tt^2 + (3-q)t^3))\Gamma(4-\gamma)}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right) \\ & \times \int_0^T (T-s)^{\alpha-\gamma-1} v_n(s) ds \end{aligned}$$

for all $t \in I$. Since F has compact values, there is a subsequence of $\{v_n\}$ that converges to v in $L^1(I, \mathbb{R})$. Thus, $v \in S_{F,x}$ and

$$\begin{aligned} u_n(t) \rightarrow u(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} v(s) ds \\ & + \frac{[bT - (a+b)t]\Gamma(2-p)}{(a+b)\Gamma(\alpha-p)T^{1-p}} \int_0^T (T-s)^{\alpha-p-1} v(s) ds \\ & - \frac{[bpT^2 - (a+b)(2Tt - (2-p)t^2)]\Gamma(3-q)}{2(a+b)(2-p)\Gamma(\alpha-q)T^{2-q}} \int_0^T (T-s)^{\alpha-q-1} v(s) ds \\ & - \left(\frac{[b(-6(q-p) + (2-p)(3-p)q)T^3]}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right. \\ & \left. + \frac{(a+b)(6(q-p)T^2t + (2-p)(3-p)(-3Tt^2 + (3-q)t^3))\Gamma(4-\gamma)}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right) \\ & \times \int_0^T (T-s)^{\alpha-\gamma-1} v(s) ds \end{aligned}$$

for all $t \in I$. This implies that $u \in \Omega(x)$. Next, we show that there exists $\Theta < 1$ such that

$$d_H(\Omega(z), \Omega(\tilde{z})) \leq \Theta \|z - \tilde{z}\|$$

for all $z, \tilde{z} \in X$. Let $z, \tilde{z} \in X$ and $g_1 \in \Omega(z)$. Choose $f_1 \in S_{F,z}$ such that

$$\begin{aligned} g_1(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_1(s) ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f_1(s) ds \\ & + \frac{[bT - (a+b)t]\Gamma(2-p)}{(a+b)\Gamma(\alpha-p)T^{1-p}} \int_0^T (T-s)^{\alpha-p-1} f_1(s) ds \\ & - \frac{[bpT^2 - (a+b)(2Tt - (2-p)t^2)]\Gamma(3-q)}{2(a+b)(2-p)\Gamma(\alpha-q)T^{2-q}} \int_0^T (T-s)^{\alpha-q-1} f_1(s) ds \\ & - \left(\frac{[b(-6(q-p) + (2-p)(3-p)q)T^3]}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right. \end{aligned}$$

$$+ \frac{(a+b)(6(q-p)T^2t + (2-p)(3-p)(-3Tt^2 + (3-q)t^3))\Gamma(4-\gamma)}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \Bigg) \\ \times \int_0^T (T-s)^{\alpha-\gamma-1} f_1(s) ds$$

for all $t \in I$. On the other hand we have

$$d_H(\tilde{F}(t, z(t)), \tilde{F}(t, \tilde{z}(t))) \\ \leq \left[k_1(t) + k_2(t) + k_3(t) + k_4(t) \right. \\ + k_5(t)\gamma_0 \left(\eta_{11} + \eta_{21} + \eta_{31} + \eta_{41} + \eta_{51} \frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} + \eta_{61} \frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + \eta_{71} \frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)} \right) \\ + k_6(t)\lambda_0 \left(\eta_{12} + \eta_{22} + \eta_{32} + \eta_{42} + \eta_{52} \frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} + \eta_{62} \frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + \eta_{72} \frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)} \right) \\ \left. + \sum_{i=1}^m m_i(t) \frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} + \sum_{j=1}^{m'} m'_j(t) \frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} + \sum_{k=1}^{m''} m''_k(t) \frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)} \right] \|z - \tilde{z}\|$$

for almost all $t \in I$. Hence, there exists $w_t \in \tilde{F}(t, \tilde{z}(t))$ such that

$$|f_1(t) - w_t| \\ \leq \left[k_1(t) + k_2(t) + k_3(t) + k_4(t) \right. \\ + k_5(t)\gamma_0 \left(\eta_{11} + \eta_{21} + \eta_{31} + \eta_{41} + \eta_{51} \frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} + \eta_{61} \frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + \eta_{71} \frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)} \right) \\ + k_6(t)\lambda_0 \left(\eta_{12} + \eta_{22} + \eta_{32} + \eta_{42} + \eta_{52} \frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} + \eta_{62} \frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + \eta_{72} \frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)} \right) \\ \left. + \sum_{i=1}^m m_i(t) \frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} + \sum_{j=1}^{m'} m'_j(t) \frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} + \sum_{k=1}^{m''} m''_k(t) \frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)} \right] \|z - \tilde{z}\| := M_t$$

for almost all $t \in I$. Define $V : I \rightarrow P(\mathbb{R})$ by $V(t) = \{u \in \mathbb{R} : |f_1(t) - u| \leq M_t\}$ for all $t \in I$. By Theorem III-41 in [47], it follows that V is measurable. Since the multivalued operator $t \mapsto V(t) \cap \tilde{F}(t, \tilde{z}(t))$ is measurable (Proposition III-4 in [47]), there exists a function $f_2 \in S_{F, \tilde{z}}$ such that

$$|f_1(t) - f_2(t)| \\ \leq \left[k_1(t) + k_2(t) + k_3(t) + k_4(t) \right. \\ + k_5(t)\gamma_0 \left(\eta_{11} + \eta_{21} + \eta_{31} + \eta_{41} + \eta_{51} \frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} + \eta_{61} \frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + \eta_{71} \frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)} \right) \\ + k_6(t)\lambda_0 \left(\eta_{12} + \eta_{22} + \eta_{32} + \eta_{42} + \eta_{52} \frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} + \eta_{62} \frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + \eta_{72} \frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)} \right) \\ \left. + \sum_{i=1}^m m_i(t) \frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} + \sum_{j=1}^{m'} m'_j(t) \frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} + \sum_{k=1}^{m''} m''_k(t) \frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)} \right] \|z - \tilde{z}\|$$

for almost all $t \in I$. Define

$$\begin{aligned} g_2(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_2(s) ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f_2(s) ds \\ & + \frac{[bT - (a+b)t]\Gamma(2-p)}{(a+b)\Gamma(\alpha-p)T^{1-p}} \int_0^T (T-s)^{\alpha-p-1} f_2(s) ds \\ & - \frac{[bpT^2 - (a+b)(2Tt - (2-p)t^2)]\Gamma(3-q)}{2(a+b)(2-p)\Gamma(\alpha-q)T^{2-q}} \int_0^T (T-s)^{\alpha-q-1} f_2(s) ds \\ & - \left(\frac{[b(-6(q-p) + (2-p)(3-p)q)T^3]}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right. \\ & \left. + \frac{(a+b)(6(q-p)T^2t + (2-p)(3-p)(-3Tt^2 + (3-q)t^3))\Gamma(4-\gamma)}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right) \\ & \times \int_0^T (T-s)^{\alpha-\gamma-1} f_2(s) ds \end{aligned}$$

for all $t \in I$. Then we have

$$\begin{aligned} \|g_1 - g_2\| &= \sup_{t \in I} |g_1(t) - g_2(t)| + \sup_{t \in I} |g_1'(t) - g_2'(t)| + \sup_{t \in I} |g_1''(t) - g_2''(t)| + \sup_{t \in I} |g_1'''(t) - g_2'''(t)| \\ &\leq M_2 \left[\|k_1\|_1 + \|k_2\|_1 + \|k_3\|_1 + \|k_4\|_1 \right. \\ &\quad + \|k_5\|_1 \gamma_0 \left(\eta_{11} + \eta_{21} + \eta_{31} + \eta_{41} + \eta_{51} \frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} + \eta_{61} \frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + \eta_{71} \frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)} \right) \\ &\quad + \|k_6\|_1 \lambda_0 \left(\eta_{12} + \eta_{22} + \eta_{32} + \eta_{42} + \eta_{52} \frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} + \eta_{62} \frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + \eta_{72} \frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)} \right) \\ &\quad \left. + \sum_{i=1}^m \|m_i\|_1 \frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} + \sum_{j=1}^{m'} \|m_j'\|_1 \frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} + \sum_{k=1}^{m''} \|m_k''\|_1 \frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)} \right] \|z - \tilde{z}\| \\ &= \Theta \|z - \tilde{z}\|. \end{aligned}$$

Further, interchanging the roles of z and \tilde{z} , we get

$$d_H(\Omega(z), \Omega(\tilde{z})) \leq \Theta \|z - \tilde{z}\|.$$

Since $\Theta < 1$, Ω is a contraction and so by Theorem 4.4, Ω has a fixed point which corresponds to a solution of problem (1.2)-(1.3). \square

5 Examples

This section is devoted to the illustration of Theorems 4.5 and 4.7.

Example 5.1 Consider the fractional differential inclusion

$$\begin{aligned} {}^c D^{\frac{9}{2}} x(t) \in & F(t, x(t), x'(t), x''(t), x'''(t), \varphi x(t), \\ & {}^c D^{\frac{2}{5}} x(t), {}^c D^{\frac{4}{3}} x(t), {}^c D^{\frac{7}{3}} x(t), {}^c D^{\frac{13}{6}} x(t)), \quad t \in [0, 1] \end{aligned} \quad (5.1)$$

supplemented with the boundary conditions $x^{(4)}(0) = 0$, $\frac{1}{5}x(0) + \frac{1}{2}x(1) = 0$, ${}^c D^{\frac{1}{3}}x(0) = -{}^c D^{\frac{1}{3}}x(1)$, ${}^c D^{\frac{7}{5}}x(0) = -{}^c D^{\frac{7}{5}}x(1)$, and ${}^c D^{\frac{9}{4}}x(0) = -{}^c D^{\frac{9}{4}}x(1)$ where

$$\begin{aligned} \varphi x(t) = & \int_0^t \frac{e^{-(s-t)/2}}{900} \left[\frac{e^{-\pi t}}{16(1+t^2)} + \frac{2x(s)}{517(1+t)(2+\sin x(s))} + \frac{5e^{-st}x'(s)}{327(s^2+7)} \right. \\ & + \frac{\sqrt[3]{\pi}x'(s)x''(s)}{984(1+|x'(s)|)} + \frac{3e^{-\cos^2 x(s)}x'''(s)}{895(s^2+1)} + \frac{\sin x(s){}^c D^{\frac{1}{4}}x(s)}{1,875\sqrt{1+|{}^c D^{\frac{1}{4}}x(s)|+|{}^c D^{\frac{5}{3}}x(s)|}} \\ & \left. + \frac{e^{-\pi} \cos^2 x(s){}^c D^{\frac{5}{3}}x(s)}{3,641(s^2+2s+1)} + \frac{{}^c D^{\frac{11}{5}}x(s)}{6,795(1+|x'(s)|)} \right] ds. \end{aligned}$$

Here $\alpha = \frac{9}{2}$, $\mu_1 = \frac{2}{5}$, $\nu_1 = \frac{4}{3}$, $\xi_1 = \frac{7}{3}$, $\xi_2 = \frac{13}{6}$, $p = \frac{1}{3}$, $q = \frac{7}{5}$, $\gamma = \frac{9}{4}$, $a = \frac{1}{5}$, $b = \frac{1}{2}$, $\delta_1 = \frac{1}{4}$, $\beta_1 = \frac{5}{3}$, $\theta_1 = \frac{11}{5}$, $c_{01} = \frac{1}{16}$, $l_{11} = \frac{2}{517}$, $l_{21} = \frac{5}{2,289}$, $l_{31} = \frac{\sqrt[3]{\pi}}{984}$, $l_{41} = \frac{3}{895}$, $l_{51} = \frac{1}{1,875}$, $l_{61} = \frac{e^{-\pi}}{3,641}$, $l_{71} = \frac{1}{6,795}$, $\gamma_0 = \frac{\sqrt{e}-1}{450}$. Define the multifunction $F : [0, 1] \times \mathbb{R}^9 \rightarrow P(\mathbb{R})$ by

$$\begin{aligned} F(t, x_1, x_2, x_3, x_4, x_5, y_1, z_1, w_1, w_2) = & \left\{ u \in \mathbb{R} : -\frac{|x_1|^3}{4(3+|x_1|^3)} - \frac{e^{-\pi t}}{5} \sin^2 x_2 - \frac{|x_5|}{13(4+\sin^2 x_1)^2} - \frac{9|y_1|}{10(4+|y_1|)} \right. \\ & - \frac{t^3}{9} \cos^2 z_1 - \frac{e^{-\pi t^2}}{81(t^4+3)} |w_1| \\ & - \frac{t^{\frac{1}{3}}}{20(1+|w_2|)} \leq u \leq \frac{1}{3} e^{-|x_1|} + \frac{7e^{-\pi t^2}}{18(1+tx_2^2)} + \frac{|x_4|}{10(1+|x_4|)} \\ & + \frac{t^{\frac{3}{2}}|x_5+\sin x_5|}{119} + \frac{e^{-t}}{t^4+2} \sin^4 y_1 + \frac{e^{-\frac{3}{2}t}}{5\sqrt{1+|z_1|^{\frac{5}{2}}}} \\ & \left. + \frac{t}{135(t^2+1)} \left| w_1 + \frac{x_3}{1+|x_3|} \right| + \frac{31|w_2|^3}{140(1+|w_2|^3)} + \frac{3}{2} \right\} \end{aligned}$$

and note that

$$\begin{aligned} \|F(t, x_1, x_2, x_3, x_4, x_5, y_1, z_1, w_1, w_2)\|_p = & \sup\{|v| : v \in F(t, x_1, x_2, x_3, x_4, x_5, y_1, z_1, w_1, w_2)\} \\ \leq & \frac{4,591}{1,260} + \frac{1}{119}(|x_5|+1) + \frac{1}{135}(|w_1|+1) \end{aligned}$$

for all $t \in [0, 1]$ and $x_1, x_2, x_3, x_4, x_5, y_1, z_1, w_1, w_2 \in \mathbb{R}$. It is clear that F has convex and compact values and is of Carathéodory type. Let $q_1(t) = 1$, $\varphi_1(|x_1|) = \frac{4}{5}$, $q_2(t) = 1$, $\varphi_2(|x_2|) = \frac{1}{2}$, $q_3(t) = 1$, $\varphi_3(|x_3|) = \frac{1}{5}$, $q_4(t) = 1$, $\varphi_4(|x_4|) = \frac{1}{2}$, $q_5(t) = 1$, $\varphi_5(|x_5|) = \frac{1}{119}(|x_5|+1)$, $\rho_1(t) = 1$, $\psi_1(|y_1|) = \frac{8}{9}$, $\rho'_1(t) = 1$, $\psi'_1(|z_1|) = \frac{1}{60}$, $\rho''_1(t) = 1$, $\psi''_1(|w_1|) = \frac{1}{135}(|w_1|+1)$, $\rho''_2(t) = 1$, and $\psi''_2(|w_2|) = \frac{31}{42}$ for all $t \in [0, 1]$ and $x_1, x_2, x_3, x_4, x_5, y_1, z_1, w_1, w_2 \in \mathbb{R}$. Hence,

$$\begin{aligned} \|F(t, x_1, x_2, x_3, x_4, x_5, y_1, z_1, w_1, w_2)\|_p = & \sup\{|v| : v \in F(t, x_1, x_2, x_3, x_4, x_5, y_1, z_1, w_1, w_2)\} \\ \leq & q_1(t)\varphi_1(|x_1|) + q_2(t)\varphi_2(|x_2|) + q_3(t)\varphi_3(|x_3|) + q_4(t)\varphi_4(|x_4|) \\ & + q_5(t)\varphi_5(|x_5|) + \rho_1(t)\psi_1(|y_1|) + \rho'_1(t)\psi'_1(|z_1|) + \rho''_1(t)\psi''_1(|w_1|) + \rho''_2(t)\psi''_2(|w_2|) \end{aligned}$$

for all $t \in [0, 1]$ and $x_1, x_2, x_3, x_4, x_5, y_1, z_1, w_1, w_2 \in \mathbb{R}$. With the given values, it is found that $M_2 \cong 8.609986787$. Letting $D > 33.90317546$, all the conditions of Theorem 4.5 hold and consequently problem (5.1) has at least one solution.

Example 5.2 Consider the fractional differential inclusion

$$\begin{aligned} {}^c D^{\frac{16}{3}} x(t) &\in F(t, x(t), x'(t), x''(t), x'''(t), \varphi x(t), \\ & {}^c D^{\frac{1}{19}} x(t), {}^c D^{\frac{2}{15}} x(t), {}^c D^{\frac{20}{17}} x(t), {}^c D^{\frac{11}{5}} x(t)), \quad t \in [0, 1] \end{aligned} \quad (5.2)$$

with the boundary value problems $x^{(4)}(0) = x^{(5)}(0) = 0$, $x(0) - 3x(1) = 0$, ${}^c D^{\frac{3}{20}} x(0) = -{}^c D^{\frac{3}{20}} x(1)$, ${}^c D^{\frac{15}{14}} x(0) = -{}^c D^{\frac{15}{14}} x(1)$, and ${}^c D^{\frac{19}{9}} x(0) = -{}^c D^{\frac{19}{9}} x(1)$, where

$$\begin{aligned} \varphi x(t) &= \int_0^t \frac{(s-t)^2 e^{-(s-t)^3}}{1,350} \left[\frac{t^3 \cos^2 t}{e^{\pi t}(1+t^2)} + \frac{|x(s) + x'(s) + {}^c D^{\frac{1}{12}} x(s)|}{9,416\pi(1 + |x(s) + x'(s) + {}^c D^{\frac{1}{12}} x(s)|)} \right. \\ &+ \frac{t^3 \sin^2 t \cos s}{759(36\sqrt{\pi} + e^{3s})} \arctan\left(\frac{3}{2} + \frac{|x''(s) + {}^c D^{\frac{7}{4}} x(s)|}{1 + |x''(s) + {}^c D^{\frac{7}{4}} x(s)|}\right) \\ &\left. + \frac{e^{st}}{8,190(1 + e^{st})} \cos x'''(s) + \frac{s^2 e^{-\pi s^3} |{}^c D^{\frac{41}{20}} x(s)|}{(1,200 + \arcsin(\frac{1}{3})e^{3t^2})(1 + |{}^c D^{\frac{41}{20}} x(s)|)} \right] ds. \end{aligned}$$

Here $\alpha = \frac{16}{3}$, $\mu_1 = \frac{1}{19}$, $\mu_2 = \frac{2}{15}$, $\nu_1 = \frac{20}{17}$, $\xi_1 = \frac{11}{5}$, $p = \frac{3}{20}$, $q = \frac{15}{14}$, $\gamma = \frac{19}{9}$, $a = 1$, $b = -3$, $\delta_1 = \frac{1}{12}$, $\beta_1 = \frac{7}{4}$, $\theta_1 = \frac{41}{20}$, $n_{11} = n_{21} = n_{51} = \frac{1}{9,416\pi}$, $n_{31} = n_{61} = \frac{1}{759(36\sqrt{\pi}+1)}$, $n_{41} = \frac{1}{8,190}$, $n_{71} = \frac{1}{1,200+\arcsin(\frac{1}{3})}$, $\gamma_0 = \frac{e-1}{4,050}$. We define the multifunction $F: [0, 1] \times \mathbb{R}^9 \rightarrow P(\mathbb{R})$ as

$$\begin{aligned} &F(t, x_1, x_2, x_3, x_4, x_5, y_1, y_2, z_1, w_1) \\ &= \left[-\frac{e^{-\pi t}}{1+t^2} - \arctan\left(1 + \frac{e^t |x_1 + x_2 + x_3 + x_4|}{9,600(\frac{1}{3} + e^t)(1 + |x_1 + x_2 + x_3 + x_4|)}\right), \cos^2 t \right] \\ &\cup \left[7 + \frac{\sin \pi t}{\sqrt{2+t^3}}, 20(t^3 + 1) + \cos\left(t^3 + \frac{e^t |y_1 + y_2|}{8,719\pi(1 + |y_1 + y_2|)}\right) \right. \\ &\left. + \frac{|z_1 + w_1|}{5,170(9+t)^4(1 + |z_1 + w_1|)} \right] \\ &\cup \left[\frac{3}{2}, \frac{|x_5|}{7,491(t+25)^5(1 + |x_5|)} + t^2 + \frac{5}{2} \right] \end{aligned}$$

for all $t \in [0, 1]$ and $x_1, x_2, x_3, x_4, x_5, y_1, y_2, z_1, w_1 \in \mathbb{R}$. It is clear that F has compact values and

$$\begin{aligned} &d_H(F(t, x_1, x_2, x_3, x_4, x_5, y_1, y_2, z_1, w_1), F(t, x'_1, x'_2, x'_3, x'_4, x'_5, y'_1, y'_2, z'_1, w'_1)) \\ &\leq \frac{e^t}{9,600(\frac{1}{3} + e^t)} (|x_1 - x'_1| + |x_2 - x'_2| + |x_3 - x'_3| + |x_4 - x'_4|) \\ &+ \frac{1}{7,491(t+25)^5} |x_5 - x'_5| + \frac{e^t}{8,719\pi} (|y_1 - y'_1| + |y_2 - y'_2|) \\ &+ \frac{1}{5,170(9+t)^4} (|z_1 - z'_1| + |w_1 - w'_1|) \end{aligned}$$

for all $t \in [0, 1]$ and $x_1, x_2, x_3, x_4, x_5, x'_1, x'_2, x'_3, x'_4, x'_5, y_1, y_2, y'_1, y'_2, z_1, z'_1, w_1, w'_1 \in \mathbb{R}$. Fix $k_1(t) = k_2(t) = k_3(t) = k_4(t) = \frac{e^t}{9,600(\frac{1}{3} + e^t)}$, $k_5(t) = \frac{1}{7,491(t+25)^5}$, $m_1(t) = m_2(t) = \frac{e^t}{8,719\pi}$, $m'_1(t) = m''_1(t) = \frac{1}{5,170(9+t)^4}$. As in the previous example, it is found that $M_2 \cong 4.046590862$ and

$$\begin{aligned} \Theta = M_2 & \left[\|k_1\|_1 + \|k_2\|_1 + \|k_3\|_1 + \|k_4\|_1 \right. \\ & + \|k_5\|_1 \gamma_0 \left(n_{11} + n_{21} + n_{31} + n_{41} + n_{51} \frac{1}{\Gamma(2 - \delta_1)} + n_{61} \frac{1}{\Gamma(3 - \beta_1)} + n_{71} \frac{1}{\Gamma(3 - \theta_1)} \right) \\ & \left. + \|m_1\|_1 \frac{1}{\Gamma(2 - \mu_1)} + \|m_2\|_1 \frac{1}{\Gamma(2 - \mu_2)} + \|m'_1\|_1 \frac{1}{\Gamma(3 - v_1)} + \|m''_1\|_1 \frac{1}{\Gamma(4 - \xi_1)} \right] \\ & \cong 1.922566928 \times 10^{-3} < 1. \end{aligned}$$

As all the conditions of Theorem 4.7 are satisfied, the inclusion problem (5.2) has at least one solution.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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